keenonmaths

The Collatz conjecture

Nihil sub sole novum

The 3n+1 problem nobody can solve

Simple but unsolved · No. 1

written by **Nicola Maglione** — keenonmaths

@keenonmaths

October 2026

Contents

Introduction. A children's puzzle, unsolved for almost a century ●○○ 1

1 The statement ●●○ 3

2 What a little can prove ●●○ 4

3 Why we believe it, and why that is not enough ●●○ 6

4 What we know today ●●○ 10

5 Your turn ●●○ 10

Appendix A. Proof of Theorem 3.4 ●●● 12

Bibliography 13

●○○ For everyone
●●○ High school and up
●●● University

Introduction. A children's puzzle, unsolved for almost a century ●○○

Think of a positive whole number, any one at all. Now follow two rules:

- if the number is **even**, halve it;
- if it is **odd**, triple it and add 1.

Take the result and start again, and keep going.

Let's start from 6. It is even, so it becomes 3. 3 is odd: tripled it makes 9, plus 1 makes 10. Then 5, then 16, then 8, 4, 2 and finally 1. Once you reach 1, the rules go round in a circle: 1 becomes 4, then 2, then 1 again. That is the end of the line, and that is where we stop.

With 7 the road is longer: it goes through 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, and only then reaches 1, after 16 steps.

27 has a surprise in store. It looks like a quiet number, yet its sequence climbs for dozens of steps, falls, climbs again, and reaches a peak of **9232**, more than three hundred times its starting point. Then it collapses, and reaches 1 after **111 steps**. Its neighbour 28, on the other hand, gets there in 18.

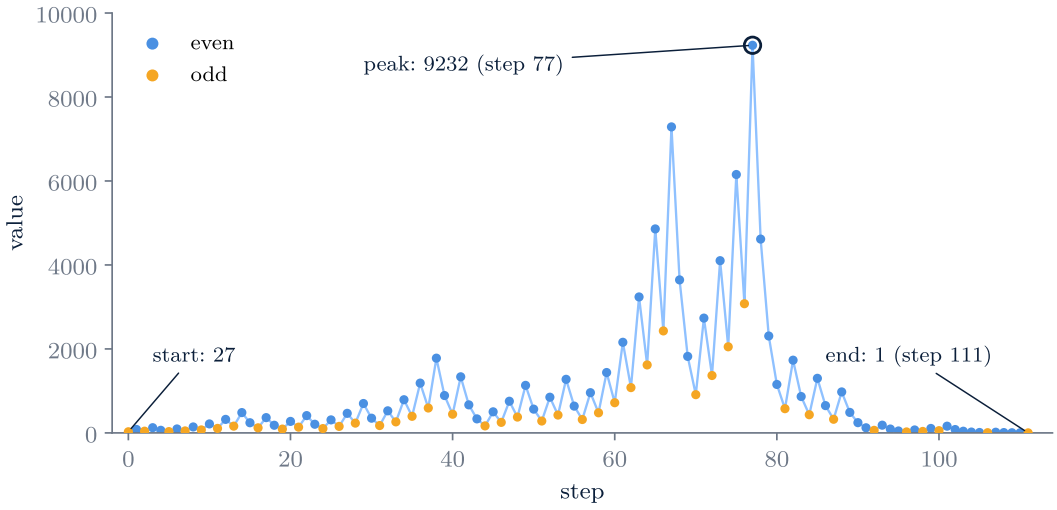

Figure 1. The path of the number 27: 111 steps, with a peak of 9232 at step 77.

Try any number you like: sooner or later you will reach 1. Every number anyone has ever tried has got there. The question is very simple:

The question

Starting from any positive whole number, do you always reach 1?

Nobody knows the answer. This is the **Collatz conjecture**: a *conjecture* is a statement believed to be true that nobody has yet managed to prove.

A problem with many names

Its origins are not entirely clear, because for decades the problem travelled by word of mouth rather than in journals. It is traditionally attributed to Lothar Collatz, a German mathematician at the University of Hamburg: in the 1930s, as a student, he studied functions of this kind, and one of his notebooks, dated 1 July 1932, contains a closely related one. Collatz never published these problems, but he discussed them with colleagues at the 1950 International Congress of Mathematicians in Cambridge, Massachusetts. Another mathematician, Bryan Thwaites, claimed to have discovered it independently in 1952.

From there the problem travelled from university to university, picking up a new name each time. Helmut Hasse, a colleague of Collatz, talked about it with many people and proposed the name **Syracuse problem** during a visit to Syracuse University, in New York State. Around 1960 Shizuo Kakutani circulated it at Yale: he said that for about a month everybody there worked on it with no result, and that at Chicago people joked that the problem was part of a conspiracy to slow down mathematical research in the United States. Stanisław Ulam, the Manhattan Project mathematician, spread it at Los Alamos. So today it goes by the Collatz problem, the Syracuse problem, Kakutani's problem, Ulam's problem, or simply the $3n + 1$ **problem**.

Several mathematicians have offered money for a solution: 50 dollars from Harold Coxeter in 1970, 500 dollars from Paul Erdős. Erdős, one of the most prolific mathematicians of the twentieth century, is reported to have said: "Mathematics is not yet ready for such problems"

[Lagarias 1985].

What we know today

Computers have checked the conjecture on enormous numbers. In 2025 David Bařina, of Brno University of Technology, published a verification for all numbers up to 2^{71}, that is, more than two sextillion, spreading the calculation over thousands of processors on European supercomputers [Bařina 2025]. Not a single exception. But the whole numbers are infinite, and no computer can try them all.

On the proof side, the biggest step forward came in September 2019. Terence Tao, one of the best-known mathematicians in the world, proved that the conjecture is true "almost always", in a precise sense: almost all numbers sooner or later fall to values that are tiny compared with where they started [Tao 2022]. The result attracted attention beyond specialist circles, and experts welcomed it as the best in a very long time [Hartnett 2019]. But "almost all" is not "all": somewhere among the numbers there might still be one that escapes the rule.

The same article reports that young researchers are often advised to stay away from it: it is a problem that can turn into an obsession and cost years with nothing to show for them [Hartnett 2019].

Why should we care?

The Collatz conjecture will not help build bridges or make phones work. Yet it is a serious question. It combines two elementary operations, multiplying by 3 and dividing by 2, and shows how little we can predict about what happens when they alternate. If such a simple problem has resisted for almost a century, some idea is still missing, and the ideas born to solve problems like this often end up being useful elsewhere.

The pages that follow go into the details, with difficulty that grows a little at a time. Next to each section title, one to three filled dots show its level: ●○○ **For everyone**, ●●○ **High school and up** (for readers comfortable with the mathematics of the last two years of high school) and ●●● **University** (proofs in the language of university courses). You can stop wherever you like.

1. The statement ●●○

Definition 1.1 (Collatz map)

The *Collatz map* is the function C defined on the positive integers by

$$C(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 3n+1 & \text{if } n \text{ is odd.} \end{cases}$$

The *orbit* of n is the sequence $n,\ C(n),\ C(C(n)),\ \ldots$ obtained by applying C repeatedly. If the orbit reaches 1, the number of applications of C needed is called the *total stopping time* of n.

In this language, the number 27 has total stopping time 111 and the number 1 has total stopping time 0.

Collatz conjecture

For every positive integer n, the orbit of n reaches 1.

Remark 1.2. If the conjecture were false, there would be a number whose orbit never reaches 1. Such an orbit can behave in only two ways:

1. sooner or later it returns to a value it has already visited, and so closes up into a **cycle** other than $1 \to 4 \to 2 \to 1$;
2. it never returns to a value already visited, and so takes infinitely many distinct values: it is **unbounded**, that is, it grows beyond any limit.

Indeed, if an orbit stayed bounded, it could take only finitely many values and would be forced to repeat one; from then on each term determines the next, and the orbit repeats forever. The conjecture is therefore equivalent to two statements together: *there are no other cycles* and *there are no unbounded orbits*. Neither has been proved.

2. What a little can prove ●●○

Some cases can be settled with entirely elementary tools. The first is immediate.

Example 2.1

Powers of 2 reach 1 by halving: $2^k \to 2^{k-1} \to \cdots \to 1$, in k steps. Odd numbers that become a power of 2 after a single step also get there quickly. Since $4^k - 1$ is always divisible by 3, the numbers

$$\frac{4^k - 1}{3} = 1,\ 5,\ 21,\ 85,\ 341,\ \ldots$$

are odd integers, and satisfy $3 \cdot \dfrac{4^k - 1}{3} + 1 = 4^k$.

A more interesting question is when an orbit **drops below its starting value**. For even numbers the answer is trivial: after one step the value has been halved. For a quarter of the odd numbers it takes only a little more.

Proposition 2.2

Let $n > 1$. If n is even, or of the form $n = 4t + 1$ with t an integer, then the orbit of n drops below n in at most 3 steps.

Proof. If n is even, $C(n) = n/2 < n$. If $n = 4t+1$ with $t \geq 1$ (the case $t = 0$ gives $n = 1$, which is excluded), then n is odd and

$$4t + 1 \ \to\ 12t + 4 \ \to\ 6t + 2 \ \to\ 3t + 1,$$

where the second and third steps are divisions by 2, because $12t + 4$ and $6t + 2$ are even. Finally $3t + 1 < 4t + 1$ because $t \geq 1$. ■

Dropping below the starting value may look like a modest result. It is not: if it held for every number, the conjecture would be proved.

Proposition 2.3

Suppose that, for every $n > 1$, the orbit of n contains at least one value smaller than n. Then the Collatz conjecture is true.

Proof. We use strong induction on n. For $n = 1$ there is nothing to prove. Let $n > 1$ and assume that all numbers smaller than n have an orbit that reaches 1. By hypothesis the orbit of n hits a value $m < n$; from there on it coincides with the orbit of m, which reaches 1 by the inductive hypothesis. Hence the orbit of n also reaches 1. ■

Together, the two propositions already say something about a possible counterexample, that is, a number whose orbit never reaches 1.

Corollary 2.4

If a counterexample to the conjecture exists, the smallest counterexample n_0 is of the form $4t + 3$.

Proof. If the orbit of n_0 hit a value $m < n_0$, that value would not be a counterexample, by the minimality of n_0: the orbit of m would reach 1, and with it the orbit of n_0. So the orbit of n_0 never drops below n_0. By Proposition 2.2, n_0 is neither even nor of the form $4t + 1$; moreover $n_0 \neq 1$. The only option left is $n_0 = 4t + 3$. ■

Subtle case 2.5 (Why the case $4t + 3$ resists)

For $n = 4t + 3$ the first steps give

$$4t + 3 \;\to\; 12t + 10 \;\to\; 6t + 5 \;\to\; 18t + 16 \;\to\; 9t + 8,$$

and $9t + 8$ is **greater** than $4t + 3$. The next step depends on the parity of t, which we do not know. We can then split into subcases, looking at the remainder on division by 8, by 16, by 32 and so on. For example, numbers of the form $16t + 3$ drop below their starting value in 6 steps:

$$16t + 3 \to 48t + 10 \to 24t + 5 \to 72t + 16 \to 36t + 8 \to 18t + 4 \to 9t + 2.$$

Each time some of the cases are settled, but some always remain open. The table shows, for a few powers of 2, the percentage of residue classes for which the steps guaranteed by the remainder are **not enough** to prove the descent:

Remainder modulo	4	16	32	256	2^{10}	2^{16}	2^{20}
Classes still open	25%	18.75%	12.5%	7.42%	6.25%	3.23%	2.61%

The percentage decreases, but slowly, and never reaches zero: there is always at least one open class, that of the numbers of the form $2^k t + (2^k - 1)$ (see Remark 2.6).

Remark 2.6. The number $2^k - 1$ has an orbit that, at first, does nothing but climb. Indeed, if $m = a \cdot 2^j - 1$ with $j \geq 1$, then m is odd and

$$3m + 1 = 3a \cdot 2^j - 2, \qquad \frac{3m+1}{2} = 3a \cdot 2^{j-1} - 1.$$

So each climb "uses up" one factor of 2 and adds a factor of 3, and the result is odd again as long as some factor of 2 is left to use up. Starting from $31 = 2^5 - 1$, the first odd numbers in the orbit are

$$31,\ 47,\ 71,\ 107,\ 161,$$

each greater than the previous one; only afterwards does the orbit reach $242 = 3^5 - 1$, the first even value after which it is halved. (The orbit of 31 still reaches 1, after 106 steps and a peak of 9232.) However large k is chosen, there is always a class of numbers whose orbit climbs during the first steps: this is why no splitting into finitely many cases can settle the problem.

3. Why we believe it, and why that is not enough ●●○

After an odd step, the number $3n + 1$ is always even, and the next step divides it by 2. It is therefore convenient to group the two steps into one and study the **compressed map**

$$T(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (3n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

The map T is different from C: it produces shorter orbits, because it skips the intermediate values $3n + 1$, which is why it is called the *reduced* or *compressed* map. For example, the orbit of 7 under T is

$$7 \to 11 \to 17 \to 26 \to 13 \to 20 \to 10 \to 5 \to 8 \to 4 \to 2 \to 1,$$

that is, the one from the Introduction without the values 22, 34, 52, 40, 16. The two conjectures, however, are **equivalent**: the orbit of n under T is the orbit under C without the numbers of the form $3m + 1$ with m odd, and the number 1 is not of this form; so one reaches 1 if and only if the other does.

A step of T on an odd number multiplies the value by roughly $3/2$; on an even number it multiplies it by $1/2$. The fate of an orbit therefore depends on the sequence of evens and odds met along the way. Can this sequence be predicted just by looking at the starting number? Partly, yes, and the way it can be done is surprising.

Definition 3.1 (Parity sequence; residue class)
Let n be a positive integer and $k \geq 1$. We write $T^j(n)$ for the number obtained by applying the map T j times, starting from n. The *parity sequence of length* k of the orbit of n under T is the sequence of k letters obtained by writing, for each of the terms

$$n,\ T(n),\ T^2(n),\ \ldots,\ T^{k-1}(n),$$

an **E** if the term is even and an **O** if it is odd. Two integers belong to the same *residue class modulo* m if they leave the same remainder on division by m. There are m residue classes modulo m, one for each possible remainder $0,\ 1,\ \ldots,\ m - 1$.

For example, the orbit of 7 under T begins with 7, 11, 17, all odd: its parity sequence of length 3 is **OOO**. Modulo 4 there are four residue classes: the numbers of the form $4t$ (remainder 0), $4t+1$, $4t+2$ and $4t+3$.

Example 3.2 (Two parities, four classes)

Let us compute the parity sequence of length 2, that is, the parities of n and of $T(n)$, for each class modulo 4:

Class of n	n is…	$T(n)$	$T(n)$ is…	Sequence
$4t$	even	$2t$	even	**EE**
$4t+1$	odd	$6t+2$	even	**OE**
$4t+2$	even	$2t+1$	odd	**EO**
$4t+3$	odd	$6t+5$	odd	**OO**

Two facts stand out. First: the sequence depends only on the class, not on t; for example, all numbers of the form $4t+3$ (3, 7, 11, 15, …) begin with **OO**. Second: there are four possible two-letter sequences (**EE**, **EO**, **OE**, **OO**), and each appears for **one and only one** class.

With three parities the same thing happens, with eight classes and eight sequences.

Example 3.3 (Three parities, eight classes)

For each class modulo 8, here are three numbers in the class and their parity sequence of length 3:

Remainder modulo 8	Numbers	Sequence
0	8, 16, 24	**EEE**
1	9, 17, 25	**OEO**
2	10, 18, 26	**EOE**
3	11, 19, 27	**OOE**
4	12, 20, 28	**EEO**
5	13, 21, 29	**OEE**
6	14, 22, 30	**EOO**
7	15, 23, 31	**OOO**

Once again, numbers in the same class have the same parity sequence, and each of the eight possible sequences appears exactly once.

The following theorem says that this always happens, whatever the number k of parities we choose to look at.

Theorem 3.4 (Terras, 1976; Everett, 1977)
Let $k \geq 1$. The parity sequence of length k of the orbit of n under T depends only on the residue class of n modulo 2^k. Moreover, each of the 2^k possible sequences of k letters **E** and **O** is the parity sequence of one and only one residue class modulo 2^k.

Idea of the proof. Two numbers have the same parity if and only if their difference is even, so it is convenient to follow the difference between two orbits. Let n and n' be two numbers, and suppose that at some step the corresponding terms of their orbits have the same parity. Then at the next step their difference has been multiplied by $\frac{1}{2}$ (if the terms are even) or by $\frac{3}{2}$ (if they are odd): either way it loses exactly one factor of 2. If n and n' have the same remainder modulo 2^k, their difference contains at least k factors of 2. So it is even, and n and n' have the same parity; at the next step the difference loses one factor of 2, but still contains at least $k-1$; and so on. In general, at step i (with $i = 0,\ 1,\ \ldots,\ k-1$) the difference between the corresponding terms of the two orbits still contains at least $k - i \geq 1$ factors of 2: it is even, so the two terms have the same parity. The two parity sequences of length k coincide. If instead the remainders are different, the difference contains $j < k$ factors of 2: after j steps it becomes odd, and at that point the two terms have different parities, so the two sequences are different. Different residue classes therefore give different sequences; since classes and sequences both number 2^k, every sequence occurs for exactly one class. The full proof is in Appendix A.

This is why the theorem matters. Choosing a number at random between 1 and 2^k means choosing at random one of the 2^k residue classes; by Theorem 3.4, this is the same as choosing at random one of the 2^k sequences of **E** and **O**, each with the same probability. That is exactly what you get by tossing a coin k times and writing **E** for heads and **O** for tails. For example, with $k = 3$ each of the eight sequences in Example 3.3 has probability $1/8$, and **O** and **E** appear in each position with probability $1/2$.

Over two typical steps, one even and one odd, the value is multiplied on average by about

$$\frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4} < 1.$$

Orbits therefore *tend to go down*: on average they lose about a quarter of their value every two steps. It is this argument that makes the conjecture so plausible.

Subtle case 3.5 (An argument on average is not a proof)
Theorem 3.4 is about the parity sequence of length k of a number chosen at random. But the conjecture concerns **every** number and its **whole** orbit. A very long orbit, or a single exceptional number, is not ruled out by an argument about averages: there may be "unlucky" numbers, like someone who gets heads a hundred times in a row. The heuristic says that counterexamples, if there are any, are extremely rare; it does not say that there are none.

The exact rules matter a great deal. Changing "+1" into "−1" is enough to change the picture completely.

Example 3.6 (The $3n-1$ variant)
With the rule: even $\to n/2$, odd $\to 3n-1$, the number 5 enters a cycle that does not contain 1:

$$5 \to 14 \to 7 \to 20 \to 10 \to 5.$$

There is also a third cycle, with 18 terms, starting from 17:

$$17 \to 50 \to 25 \to 74 \to 37 \to 110 \to 55 \to 164 \to 82 \to 41$$
$$\to 122 \to 61 \to 182 \to 91 \to 272 \to 136 \to 68 \to 34 \to 17.$$

Changing the sign of every term, the $3n-1$ variant on the positive integers is the Collatz map $3n+1$ on the **negative** integers: for example $-5 \to -14 \to -7 \to -20 \to -10 \to -5$. On the negative integers, then, the Collatz map has other cycles.

The coin-tossing heuristic works for $3n-1$ too, and yet here cycles other than the trivial one appear. Any proof of the conjecture must therefore use something specific to the "+1", not just the average balance between multiplications by 3 and divisions by 2.

There is a deeper reason for caution. A *generalized Collatz map* is obtained by fixing an integer $d \geq 2$ and choosing, for each remainder r on division by d, a rule of the form $n \mapsto a_r n + b_r$, with rational coefficients such that the result is always an integer. The Collatz map is the case $d = 2$, with the rules $n/2$ and $3n+1$. In 1972 John H. Conway showed that the maps in this family can **run any program**: the program's entire memory is encoded in a single integer, through the exponents of its prime factors, and each step of the map corresponds to a step of the program [Conway 1972]. For the details of the construction, see Conway's original paper.

Example 3.7 (A map that computes a sum) ●●○
Consider the rule: if n is even, multiply it by $\frac{3}{2}$; if it is odd, stop. Starting from $n = 2^a \cdot 3^b$, each step removes a factor 2 and adds a factor 3, until no factors of 2 are left: we reach 3^{a+b}. If we read the number as a memory holding the values a and b in its exponents, the map has computed the sum $a+b$. For example, with $a = 3$ and $b = 2$:

$$72 = 2^3 \cdot 3^2 \;\to\; 108 \;\to\; 162 \;\to\; 243 = 3^5.$$

With more rules, multiplications, comparisons and loops can be built in the same way, and hence any computation.

A negative result follows. In 1936 Alan Turing proved that there is no algorithm able to decide, for every program, whether it eventually stops: this is the *halting problem* [Turing 1936]. Since every program can be translated into a generalized Collatz map, an algorithm able to predict the behaviour of the orbits of *every* map in the family would also solve the halting problem; so no such algorithm exists. This says nothing about the $3n+1$ map in particular: it is not known whether the Collatz conjecture is undecidable, nor whether the $3n+1$ map can run programs. But it explains why we should not expect a universal method.

4. What we know today ●●○

Almost all numbers go down. Theorem 3.4 has a proven consequence: the percentage of "open" classes in the table of Subtle case 2.5 tends to zero as k grows. Terras and Everett deduced from this that **almost all** positive integers have an orbit that drops below its starting value, in the sense that the proportion of numbers up to N that do not tends to 0 as N grows [Terras 1976; Everett 1977]. This is not enough for Proposition 2.3, which requires *all* numbers.

Tao's theorem. The strongest result of this type is due to Terence Tao [Tao 2022]. Given any function f that tends to infinity, however slowly, almost all orbits sooner or later drop below $f(n)$. For example, almost all numbers n have an orbit that drops below $\log\log\log\log n$ (natural logarithms, applied four times in a row).

This function tends to infinity, as the theorem requires, but with a slowness that is hard to imagine. For $\log\log\log\log n$ to exceed 1, the number n must have more than one and a half million digits. For it to exceed 2, the number of digits of n must itself be a number with more than 700 digits. The example shows how strong the theorem is: the threshold below which almost all orbits fall can grow extremely slowly, as long as it eventually exceeds every value.

The theorem, however, says nothing about any single number, nor about any finite set of numbers, however large. "Almost all" is a property in the limit: it concerns the proportion of exceptions between 1 and N as N tends to infinity (see Remark 4.1). For any single number, the theorem guarantees nothing. Certainty about specific numbers comes only from computer checks, which go up to 2^{71}.

Once again the sticking point is "almost all": Tao himself has observed that there seems to be an enormous gap in difficulty between results that hold for almost all numbers and results that hold for all of them.

Remark 4.1 (What "almost all" means). "Almost all" does not mean "all but finitely many": an infinite set of exceptions is still possible, as long as it becomes sparser and sparser. For example, the perfect squares are infinitely many, but their proportion among the numbers from 1 to N is about $\sqrt{N}/N = 1/\sqrt{N}$, which tends to 0: so almost all numbers are **not** perfect squares.

Computer checks. The limit of $2^{71} \approx 2.36 \times 10^{21}$ reached by Bařina [Bařina 2025] is the result of algorithms that do not test one number at a time: they exploit arguments just like those of Section 2, discarding from the outset the residue classes already known to drop below their starting value. By Proposition 2.3, it is enough to check that each number drops below itself, not that it goes all the way down to 1.

5. Your turn ●●○

The problems below need only the tools in these pages. Solutions are at the end.

Problem 1. Check that the cycle $5 \to 14 \to 7 \to 20 \to 10 \to 5$ of the $3n-1$ variant (Example 3.6) really is a cycle, and show that with the same rule the number 1 enters the cycle $1 \to 2 \to 1$.

Problem 2. Prove that numbers of the form $8t+5$ drop below their starting value in 3 steps of the map C. (Hint: they are a special case of a proposition we have already seen.)

Problem 3. Consider the numbers of the form $n = 16t + 11$, with $t \geq 0$ an integer.

1. Compute the first 7 steps of the map C in terms of t and check that you reach a value still greater than n.
2. The next step depends on the parity of t. For which values of t can you conclude that the orbit of n drops below n, and after how many steps?
3. What happens in the other cases? Compare the result with Subtle case 2.5.

Problem 4. Compute the total stopping time of a number of your choice between 30 and 40, by hand or with the program below.

A few lines of Python are enough to explore with a computer:

```
def collatz(n):
    orbit = [n]
    while n != 1:
        n = n // 2 if n % 2 == 0 else 3 * n + 1
        orbit.append(n)
    return orbit

o = collatz(27)
print("steps:", len(o) - 1, "  maximum:", max(o))
```

The program prints `steps: 111  maximum: 9232`. A challenge: modify it to find the number below 10 000 with the longest total stopping time.

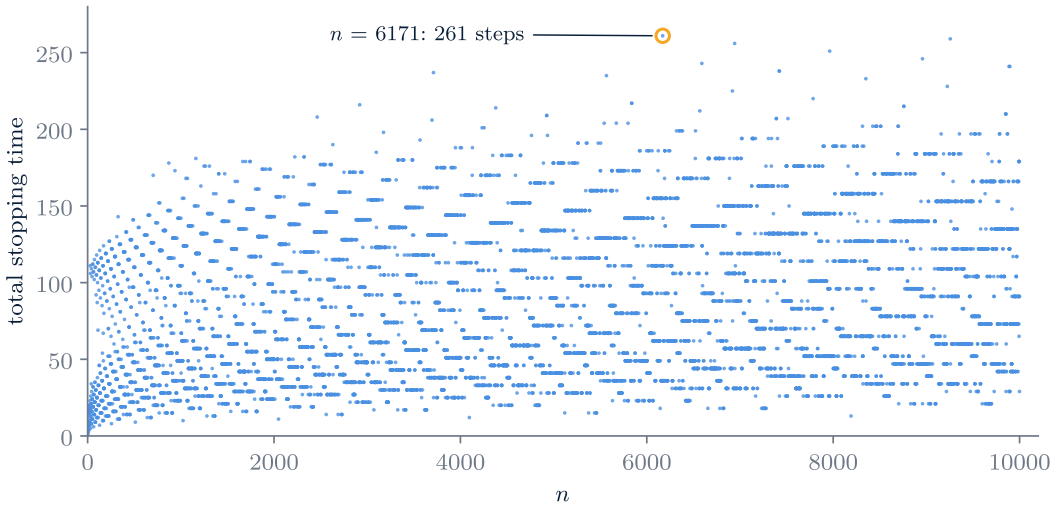

Figure 5.1. Total stopping time of the numbers from 1 to 10,000. The maximum, 261 steps, is reached by $n = 6171$. Data: OEIS A006577 [OEIS].

Solutions

Problem 1. 5 is odd: $3 \cdot 5 - 1 = 14$; then 7; $3 \cdot 7 - 1 = 20$; then 10 and 5. For 1: $3 \cdot 1 - 1 = 2$, then $2/2 = 1$.

Problem 2. $8t + 5 = 4(2t + 1) + 1$ is of the form $4s + 1$ with $s = 2t + 1 \geq 1$: Proposition

2.2 is enough.

Problem 3.

1. The first 7 steps are

   $$16t+11 \to 48t+34 \to 24t+17 \to 72t+52 \to 36t+26 \to 18t+13 \to 54t+40 \to 27t+20,$$

   and $27t+20 > 16t+11$ for every $t \geq 0$.

2. If t is even, $t = 2u$, that is $n = 32u+11$: then $27t+20 = 54u+20$ is even and the next step gives $27u+10$, which is smaller than $32u+11$. The orbit drops below n after **8 steps**.

3. If t is odd, $t = 2u+1$, that is $n = 32u+27$: then $27t+20 = 54u+47$ is odd, the next two steps give $162u+142$ and then $81u+71$, still greater than $32u+27$, and the parity at the following step depends on u. At this level the case remains **open**: the class 27 modulo 32 is indeed one of the four open classes modulo 32 in the table of Subtle case 2.5. So the numbers $16t+11$ split into two halves: for one the descent is proved, for the other we would need to split further, with no guarantee of settling the case.

Problem 4. For checking: 30 → 18 steps, 31 → 106, 32 → 5, 33 → 26, 34 → 13, 35 → 13, 36 → 21, 37 → 21, 38 → 21, 39 → 34, 40 → 8.

Appendix A. Proof of Theorem 3.4 •••

For a positive integer n we denote by $v_j(n) \in \{0,1\}$ the parity of $T^j(n)$ (0 = even, 1 = odd), and by

$$V_k(n) = \big(v_0(n),\ v_1(n),\ \ldots,\ v_{k-1}(n)\big)$$

the sequence of the first k parities.

Remark A.1. If $T^j(n)$ and $T^j(n')$ have the same parity, then

$$T^{j+1}(n) - T^{j+1}(n') = \begin{cases} \dfrac{1}{2}\big(T^j(n) - T^j(n')\big) & \text{if they are even,} \\ \dfrac{3}{2}\big(T^j(n) - T^j(n')\big) & \text{if they are odd.} \end{cases}$$

Hence, as long as the parities agree, after i steps we have

$$T^i(n) - T^i(n') = \frac{3^m}{2^i}\,(n-n'), \tag{A.1}$$

where m is the number of odd steps among the first i.

Lemma A.2
If $n \equiv n' \pmod{2^k}$, then $V_k(n) = V_k(n')$.

Proof. We argue by induction on $j = 0, 1, \ldots, k-1$, showing that $v_j(n) = v_j(n')$ and that 2^{k-j} divides $T^j(n) - T^j(n')$. For $j = 0$, 2^k divides $n - n'$ and, since $k \geq 1$, n and n' have the same parity. If the claim holds for $j < k-1$, the parities of $T^j(n)$ and $T^j(n')$ agree and, by Remark A.1, the difference $T^{j+1}(n) - T^{j+1}(n')$ is obtained from $T^j(n) - T^j(n')$ by multiplying by $1/2$ or by $3/2$; hence it is divisible by $2^{k-j-1} \geq 2$, and in particular $T^{j+1}(n)$ and $T^{j+1}(n')$ have the same parity. ■

Lemma A.3
If $n \not\equiv n' \pmod{2^k}$, then $V_k(n) \neq V_k(n')$.

Proof. Let 2^j be the highest power of 2 dividing $n - n'$: since $n \not\equiv n' \pmod{2^k}$, we have $j < k$, and $n - n' = 2^j d$ with d odd. Suppose, for a contradiction, that the first k parities agree. Then in particular the first j agree, and (A.1) with $i = j$ gives

$$T^j(n) - T^j(n') = 3^m d,$$

which is odd. Hence $T^j(n)$ and $T^j(n')$ have different parities, that is $v_j(n) \neq v_j(n')$ with $j < k$: a contradiction. ■

Proof of Theorem 3.4. By Lemma A.2, the sequence $V_k(n)$ depends only on the remainder of n modulo 2^k: this defines a function from the set of the 2^k residue classes to the set of the 2^k sequences of length k. By Lemma A.3 this function is injective. An injective map between two finite sets with the same number of elements is also surjective, hence bijective: every parity sequence occurs for exactly one residue class. ■

Bibliography

- D. Bařina, *Improved verification limit for the convergence of the Collatz conjecture*, The Journal of Supercomputing **81** (2025). https://doi.org/10.1007/s11227-025-07337-0
- J. H. Conway, *Unpredictable iterations*, Proceedings of the 1972 Number Theory Conference, University of Colorado, Boulder (1972), 49–52. Reprinted in J. C. Lagarias (ed.), *The Ultimate Challenge: The 3x+1 Problem*, American Mathematical Society, 2010. https://bookstore.ams.org/MBK/78
- C. J. Everett, *Iteration of the number-theoretic function f(2n) = n, f(2n+1) = 3n+2*, Advances in Mathematics **25** (1977), 42–45. https://doi.org/10.1016/0001-8708(77)90087-1
- K. Hartnett, *Mathematician Proves Huge Result on 'Dangerous' Problem*, Quanta Magazine, 11 December 2019. https://www.quantamagazine.org/mathematician-proves-huge-result-on-dangerous-problem-20191211/
- J. C. Lagarias, *The 3x+1 problem and its generalizations*, The American Mathematical Monthly **92** (1985), 3–23. Full text online (authorised reprint, Conference on Organic Mathematics, CMS 1997): https://www.cecm.sfu.ca/organics/papers/lagarias/paper/html/paper.html
- T. Tao, *Almost all orbits of the Collatz map attain almost bounded values*, Forum of Mathematics, Pi **10** (2022), e12. https://doi.org/10.1017/fmp.2022.8
- R. Terras, *A stopping time problem on the positive integers*, Acta Arithmetica **30** (1976), 241–252. https://doi.org/10.4064/aa-30-3-241-252
- A. M. Turing, *On computable numbers, with an application to the Entscheidungsproblem*, Proceedings of the London Mathematical Society (2) **42** (1936–37), 230–265. https://doi.org/10.1112/plms/s2-42.1.230

- OEIS Foundation, *Sequence A006577: number of halving and tripling steps to reach 1 in the 3x+1 problem.* https://oeis.org/A006577

About the author. Nicola Maglione is an electronic engineer (degree in Electronic Engineering, University of Naples, 2001), with over twenty years of experience in the semiconductor industry as a design and program manager, specialized in NAND Flash memory and high-speed interfaces, and holder of several patents in memory architectures. Alongside his professional work, he has always cultivated a passion for mathematics – particularly Analysis, Abstract Algebra, and Number Theory – and for physics, which he shares through keenonscience, his science-outreach page, and on the website keenonmaths.org.

© 2026 Nicola Maglione. Distributed under a Creative Commons Attribution – NonCommercial – NoDerivatives 4.0 International license (CC BY-NC-ND 4.0): sharing is permitted with attribution; commercial use and derivative works are not permitted.
creativecommons.org/licenses/by-nc-nd/4.0