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Continuity, Uniform Continuity, Lipschitz and Hölder Continuity

Nihil sub sole novum

Four regularity conditions: definitions, theorems, examples and counterexamples

An in-depth guide for Real Analysis I and II

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Introduction

This document is meant for anyone who wants to rigorously review the regularity notions for real-valued functions of a real variable that recur throughout calculus and real analysis courses: continuity at a point, uniform continuity, Lipschitz continuity, Hölder continuity, and – to complete the picture – absolute continuity.

The first four notions represent progressively stronger degrees of regularity. Informally,

$$\text{Lipschitz} \implies \text{Hölder} \implies \text{Uniformly continuous} \implies \text{Continuous},$$

and none of these implications reverses in general. Absolute continuity fits into this picture not as one more link in the same chain, but as a parallel branch: stronger than uniform continuity alone, implied by Lipschitz continuity, and in general not comparable with Hölder continuity (neither implies the other). The purpose of these pages is to make this hierarchy (a chain plus a fork) precise: to state the definitions with every quantifier in its right place, to prove in full the theorems that justify it (Heine–Cantor, Lagrange, the hierarchy of Hölder classes), and to provide, for each notion, both genuine examples and counterexamples that expose its limits. Particular attention is paid to the "pathological" cases that traditionally trip up anyone studying these topics for the first time, such as the Dirichlet function or the Cantor function.

Notation. Unless stated otherwise, $I \subseteq \mathbb{R}$ denotes an interval (or, more generally, a nonempty subset of $\mathbb{R}$), and $f : I \to \mathbb{R}$ denotes a generic real function defined on I, on which we will impose various regularity conditions as we go. We write $C(I)$ for the set of continuous functions on I.

1. Continuity at a Point

Definition 1.1 (Pointwise Continuity)**.** *Let $f : I \to \mathbb{R}$ and $x_0 \in I$. We say that f is **continuous at** x_0 if*

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \text{such that} \quad \forall x \in I, \quad |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

Definition 1.2 (Global Continuity)**.** *f is **continuous on** I (written $f \in C(I)$) if it is continuous at every point $x_0 \in I$.*

Remark 1.3. *The crucial point, often underestimated, is that in this definition δ may depend* both on ε and on the point x_0*: $\delta = \delta(\varepsilon, x_0)$. Changing the point, for the same ε, the δ that works can be completely different (and, as we will see, it may be forced to shrink arbitrarily). This is exactly what distinguishes pointwise continuity from uniform continuity, the subject of Section 2.*

Proposition 1.1 (Sequential Criterion for Continuity)**.** *Let $f : I \to \mathbb{R}$ and $x_0 \in I$. Then f is continuous at x_0 if and only if for every sequence $(x_n) \subset I$ with $x_n \to x_0$ we have $f(x_n) \to f(x_0)$.*

Proof. ($\Rightarrow$) Suppose f is continuous at x_0 and let $(x_n) \subset I$ with $x_n \to x_0$. Given $\varepsilon > 0$, let $\delta > 0$ be such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$. Since $x_n \to x_0$, there exists N such that $n > N \Rightarrow |x_n - x_0| < \delta$, hence $|f(x_n) - f(x_0)| < \varepsilon$ for every $n > N$: this proves $f(x_n) \to f(x_0)$.

($\Leftarrow$) We prove the contrapositive: suppose f is *not* continuous at x_0 and construct a sequence $x_n \to x_0$ with $f(x_n) \not\to f(x_0)$. Negating the definition of continuity, there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ there is $x \in I$ with $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \varepsilon_0$. Applying this with $\delta = 1/n$ for every $n \in \mathbb{N}$, we obtain $x_n \in I$ with $|x_n - x_0| < 1/n$ (hence $x_n \to x_0$) and $|f(x_n) - f(x_0)| \geq \varepsilon_0$ for every n (hence $f(x_n) \not\to f(x_0)$): this proves the claim. ∎

Example 1.1 (A Continuous Function)**.** *$f(x) = 1/x$ is continuous on $(0, 1)$. Fix $x_0 \in (0, 1)$ and $\varepsilon > 0$: we want to determine $\delta > 0$ starting from the inequality we want to guarantee,*

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{x\, x_0} < \varepsilon \qquad \Longleftrightarrow \qquad |x - x_0| < \varepsilon\, x\, x_0$$

(the last step follows simply by multiplying both sides of the inequality by $x\, x_0 > 0$). The problem is that x still appears on the right-hand side, and we don't know it until δ has been chosen: we must first eliminate it with a preliminary restriction. Impose $|x - x_0| < x_0/2$: this guarantees $x > x_0/2$, hence

$$\varepsilon\, x\, x_0 > \varepsilon\, \frac{x_0}{2}\, x_0 = \frac{\varepsilon\, x_0^2}{2}.$$

It is then enough to impose, as a second requirement, $|x - x_0| \le \dfrac{\varepsilon\, x_0^2}{2}$*: under the first restriction this implies* $|x - x_0| \le \dfrac{\varepsilon\, x_0^2}{2} < \varepsilon\, x\, x_0$*, which is what we wanted. Both requirements on* $|x - x_0|$ *must hold, so we take the smaller of the two bounds:*

$$\delta := \min\left(\frac{x_0}{2},\ \frac{\varepsilon\, x_0^2}{2}\right).$$

With this choice, $|x - x_0| < \delta \Rightarrow |1/x - 1/x_0| < \varepsilon$*, as required. Note that* δ *depends explicitly on* x_0 *(through* x_0^2*): the closer* x_0 *is to* 0*, the smaller* δ *must become, for the same* ε*.*

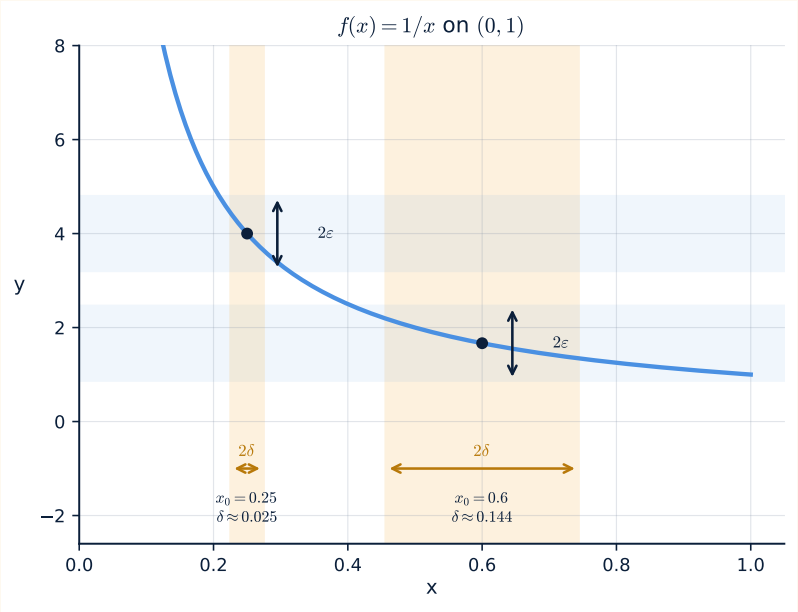


Figure 1.1. *Two different points* x_0*, with the same width* 2ε *on the vertical band (blue arrows, explicitly marked) around* $f(x_0)$*: the width* 2δ *of the corresponding horizontal band (orange arrows) must shrink the closer* x_0 *is to* 0*, in agreement with the formula* $\delta = \min(x_0/2, \varepsilon x_0^2/2)$*.*

Counterexample 1.1 (A Discontinuous Function)**.** *The step function*

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \ge 0 \end{cases}$$

is not continuous at $x_0 = 0$*: taking* $\varepsilon = 1/2$*, for every* $\delta > 0$ *there exists* $x = -\delta/2 \in (-\delta, \delta)$ *with* $|f(x) - f(0)| = 1 \ge \varepsilon$*. No* δ *can work.*

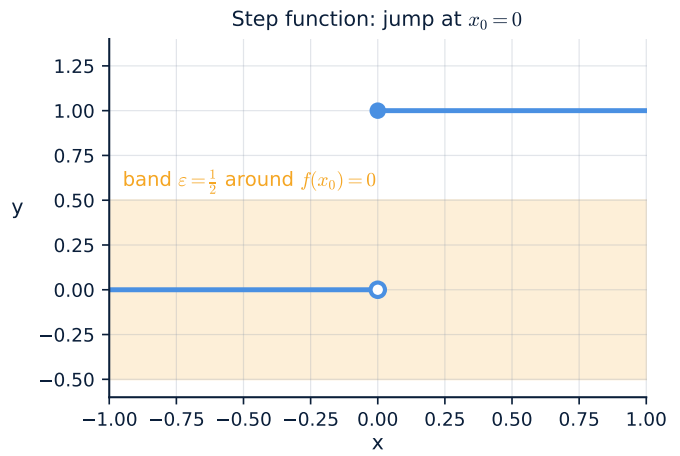


Figure 1.2. *The band $\varepsilon = 1/2$ around $f(x_0) = 0$: no neighborhood $(-\delta, \delta)$ of $x_0 = 0$, however small, manages to stay within this band.*

Subtle Case 1.1 (The Dirichlet Function: Discontinuous Everywhere)**.** *Let*

$$D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

By the density of $\mathbb{Q}$ and of $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$, every neighborhood $(x_0 - \delta, x_0 + \delta)$ of any $x_0 \in \mathbb{R}$ contains both rational and irrational points. Hence, however $\delta > 0$ is chosen, there exist x in that neighborhood with $|D(x) - D(x_0)| = 1$. Taking $\varepsilon = 1/2$ we conclude that D is not continuous at any *point of $\mathbb{R}$: it is the canonical example of an everywhere-discontinuous function, and often the first serious conceptual hurdle in the study of continuity.*

Subtle Case 1.2 (The Thomae Function: Continuous at the Irrationals)**.** *Let (the "popcorn" function)*

$$T(x) = \begin{cases} \dfrac{1}{q} & x = \dfrac{p}{q} \in \mathbb{Q},\ \gcd(p,q) = 1,\ q > 0 \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}, \qquad T(0) = 1.$$

Surprisingly, T is continuous at every *irrational point and discontinuous at* every *rational point – the exact opposite of what intuition (shaped by the Dirichlet example) might suggest.*

Continuity at irrational points. *Fix $x_0 \notin \mathbb{Q}$ and $\varepsilon > 0$, and let us show that there exists $\delta > 0$ with $|x - x_0| < \delta \Rightarrow |T(x) - T(x_0)| = T(x) < \varepsilon$ (recalling $T(x_0) = 0$ because x_0 is irrational). Set $N := \lceil 1/\varepsilon \rceil$, so that $q > N \Rightarrow 1/q < \varepsilon$.*

The key point is the following: fixing an integer $q \geq 1$, the bounded interval $[x_0 - 1, x_0 + 1]$ contains only finitely many rationals with denominator (in reduced form) equal to q. *This is because the numbers of the form p/q (as $p \in \mathbb{Z}$ varies) are equally spaced on the real line, with constant distance $1/q$ between one multiple and the next; an interval of fixed length (here, length 2) can therefore contain at most $2/(1/q) + 1 = 2q + 1$ of them – a finite number, depending on q but not on where the interval sits. For example, for $q = 2$ the*

rationals $p/2$ in the interval $[x_0-1, x_0+1]$ are at most 5 (in general 4 or 5 depending on x_0): if $x_0=\sqrt{2}\approx 1.414$, the interval is $[0.414, 2.414]$ and the multiples of $1/2$ that fall in it are $0.5,\ 1,\ 1.5,\ 2$, i.e. 4 points, consistent with the estimate.

Summing over all denominators $q=1,2,\dots,N$ (a finite number of values of q, each with finitely many associated rationals in the interval), we get that the set of rationals p/q with $q\le N$ contained in $[x_0-1, x_0+1]$ is finite. *Since x_0 is irrational, it does not coincide with any of these finitely many rationals, so it has positive distance from each; let $d>0$ be the smallest of these (finitely many, all positive) distances. Set*

$$\delta := \min(d,\, 1).$$

Then, if $|x-x_0|<\delta$: the point x belongs to $[x_0-1, x_0+1]$ and, by the choice of d, x cannot coincide with any of the rationals p/q with $q\le N$ (otherwise $|x_0-p/q|=|x_0-x|<\delta\le d$, contradicting the minimality of d). Hence x is irrational (in which case $T(x)=0<\varepsilon$) or rational with reduced denominator $q>N$ (in which case $T(x)=1/q<1/N\le\varepsilon$). In both cases $T(x)<\varepsilon$, as required: T is continuous at x_0.

Discontinuity at rational points. *Let $x_0=p/q\in\mathbb{Q}$ (reduced form), so that $T(x_0)=1/q>0$. By density of the irrationals, there exists a sequence (x_n) of irrational numbers with $x_n\to x_0$; for it $T(x_n)=0$ for every n, hence $T(x_n)\to 0\neq 1/q=T(x_0)$. By the sequential criterion for continuity (Proposition 1.1), T is not continuous at x_0.*

This example shows that "discontinuous everywhere except on a dense set" and "continuous everywhere except on a dense set" are both *realizable situations, and that geometric intuition should always be checked rigorously against the definitions.*

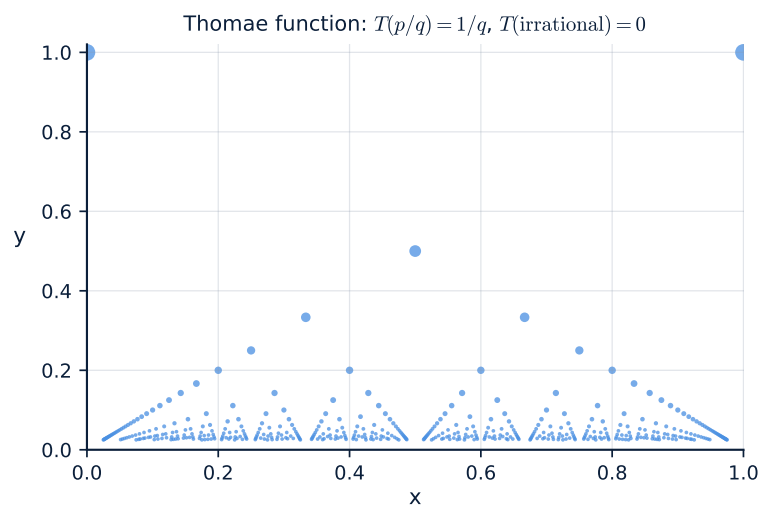


Figure 1.3. *The graph of T on $(0,1]$: every rational point p/q has height $1/q$ (smaller the larger q is), while at the irrationals the function equals 0.*

2. Uniform Continuity

Definition 2.1 (Uniform Continuity)**.** *$f : I \to \mathbb{R}$ is **uniformly continuous** on I if*

$$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 \;\; \text{such that} \;\; \forall x, y \in I, \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Remark 2.2. *The central point is that the same δ must hold* simultaneously for every pair *of points $x, y \in I$, and can no longer be adapted case by case as in pointwise continuity. This is reflected in the order of the quantifiers: writing out continuity on all of I (Definition 1.2 applied at every point) gives*

$$\forall\, x_0 \in I \;\; \forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 \;\; \forall x \in I, \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon,$$

whereas uniform continuity is

$$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 \;\; \forall\, x, y \in I, \quad |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

The difference is precisely the position of the "for every point" quantifier: in plain continuity it precedes $\exists\, \delta$ (so δ may depend on the point, $\delta = \delta(\varepsilon, x_0)$, because it is chosen after *the point has been fixed); in uniform continuity the role of the point is taken by x, y, which are quantified* after *$\exists\, \delta$ – δ must therefore be chosen before knowing which points will be considered, and it must work for all of them. Swapping the order of a $\forall$ and an $\exists$ in a logical formula is never innocuous: it is exactly what distinguishes the two notions. Every function uniformly continuous on I is continuous on I (just fix $y = x_0$ in the definition); the converse, as we will see, is false in general.*

Proposition 2.1 (Sequential Criterion for Non-Uniform Continuity)**.** *$f : I \to \mathbb{R}$ is **not** uniformly continuous on I if and only if there exist $\varepsilon_0 > 0$ and two sequences $(x_n), (y_n) \subset I$ such that*

$$|x_n - y_n| \to 0 \qquad \text{but} \qquad |f(x_n) - f(y_n)| \geq \varepsilon_0 \;\; \forall n.$$

Proof. ($\Leftarrow$) If such sequences exist, no $\delta > 0$ can satisfy the definition for $\varepsilon = \varepsilon_0$: for every δ, since $|x_n - y_n| \to 0$, there exists n with $|x_n - y_n| < \delta$, yet $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

($\Rightarrow$) If f is not uniformly continuous, negating the definition: $\exists\, \varepsilon_0 > 0$ such that $\forall\, \delta > 0$ $\exists x, y \in I$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$. Applying this with $\delta = 1/n$ for every $n \in \mathbb{N}$ we obtain $x_n, y_n \in I$ with $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$, hence $|x_n - y_n| \to 0$. ∎

Example 2.1 (A Uniformly Continuous Function)**.** *$f(x) = \sqrt{x}$ is uniformly continuous on $[0, +\infty)$. Given $\varepsilon > 0$, we want to guarantee $|\sqrt{x} - \sqrt{y}| < \varepsilon$. We first need the inequality*

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|} \qquad \forall\, x, y \geq 0,$$

which we prove right away, before using it. Assume, without loss of generality, $x \geq y \geq 0$ (the inequality is symmetric in x, y, so it suffices to treat one ordering); setting $u = \sqrt{x}$, $v = \sqrt{y}$ (so $u \geq v \geq 0$), we want to show $u - v \leq \sqrt{u^2 - v^2} = \sqrt{(u - v)(u + v)}$. If $u = v$ this is trivial; if $u > v$, squaring both (positive) sides, the claim is equivalent to

$(u-v)^2 \le (u-v)(u+v)$, *i.e., dividing by* $u-v>0$, *to* $u-v \le u+v$, *always true since* $v \ge 0$. *This proves the inequality.*

From it, it follows that it suffices to impose

$$\sqrt{|x-y|} < \varepsilon \qquad \Longleftrightarrow \qquad |x-y| < \varepsilon^2,$$

to obtain, a fortiori, $|\sqrt{x}-\sqrt{y}| < \varepsilon$. *Set* $\delta := \varepsilon^2$: *if* $|x-y| < \delta$, *then* $|\sqrt{x}-\sqrt{y}| \le \sqrt{|x-y|} < \sqrt{\delta} = \varepsilon$. *Note that* δ *does not depend on the point: the same choice works everywhere, and this is precisely what makes* f *uniformly (and not merely pointwise) continuous.*

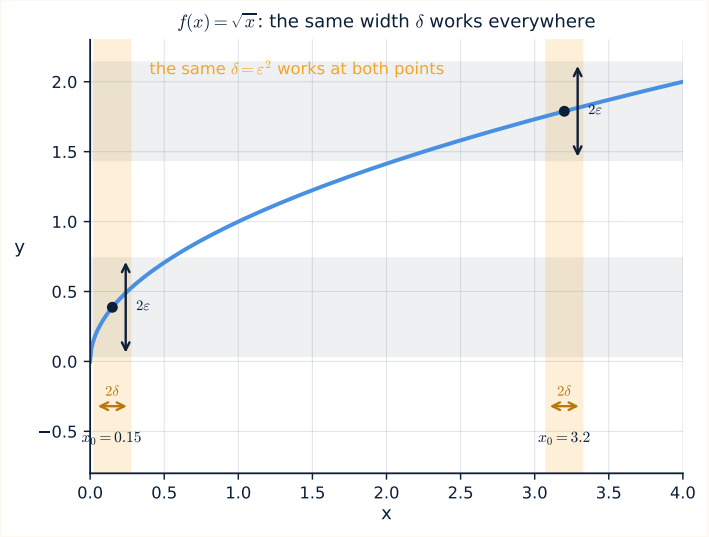


Figure 2.1. *Two different points* x_0 (0.15 *and* 3.2): *the same width* $2\delta = 2\varepsilon^2$ *(orange arrows) leads in both cases within the same width* 2ε *(grey arrows) – unlike Figure 1.1, here* δ *does not have to shrink.*

Counterexample 2.1 ($f(x) = 1/x$ Is Not Uniformly Continuous)**.** $f(x) = 1/x$ *is not uniformly continuous on* $(0,1)$. *Take* $x_n = 1/n$, $y_n = 1/(2n)$: *then* $|x_n - y_n| = 1/(2n) \to 0$, *but*

$$|f(x_n) - f(y_n)| = |n - 2n| = n \to \infty,$$

so in particular $|f(x_n)-f(y_n)| \ge 1 =: \varepsilon_0$ *eventually. By Proposition 2.1,* f *is not uniformly continuous, even though it is continuous at every point of* $(0,1)$.

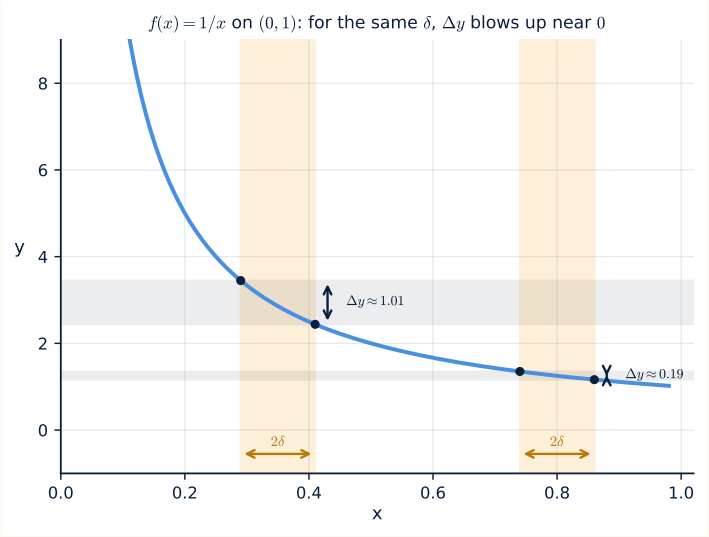


Figure 2.2. *The same width δ (orange arrows) around two points $x_0 = 0.35$ and $x_0 = 0.8$, both in $(0, 1)$: the corresponding image (grey arrows, on the y-axis) has a much larger width Δy near 0 than far from it – the same δ cannot work everywhere.*

Subtle Case 2.1 ($f(x) = x^2$ Is Continuous Everywhere but Not Uniformly Continuous on $\mathbb{R}$)**.** *It is a common mistake to think that a function continuous "everywhere" (on all of $\mathbb{R}$, not just on a bounded interval) is automatically uniformly continuous. This is not so: $f(x) = x^2$ is continuous on all of $\mathbb{R}$ but not uniformly continuous on $\mathbb{R}$. Take $x_n = n$, $y_n = n + \frac{1}{n}$: then*

$$|x_n - y_n| = \frac{1}{n} \to 0, \qquad |f(x_n) - f(y_n)| = \left| n^2 - \left(n + \tfrac{1}{n}\right)^2 \right| = 2 + \frac{1}{n^2} \geq 2 =: \varepsilon_0 \quad \forall n.$$

By Proposition 2.1, $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. The phenomenon is clear: the slope of x^2 grows without bound, so the same δ that "works well" near the origin stops working far from it. As Theorem 2.1 shows, the situation changes radically if we restrict to a compact *interval $[a, b]$: there x^2 is uniformly continuous.*

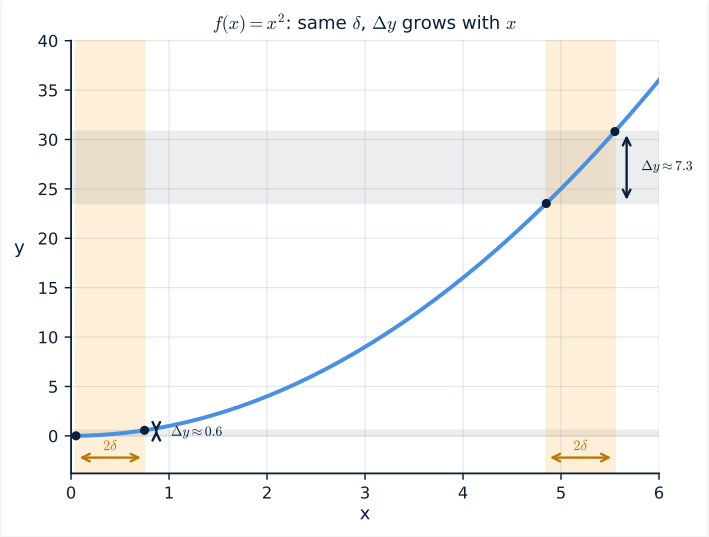


Figure 2.3. *The same width δ (orange arrows) around $x_0 = 0.4$ and $x_0 = 5.2$: the corresponding image (grey arrows) has a much larger width Δy far from the origin – here the problem is the unboundedness of the domain, not proximity to a singular point.*

Theorem 2.1 (Heine–Cantor)**.** *A function continuous on a compact set is uniformly continuous there [2].*

Proof. Let $K \subseteq \mathbb{R}$ be a compact set and $f : K \to \mathbb{R}$ a function continuous on K. Suppose, for contradiction, that f is not uniformly continuous on K. By Proposition 2.1, there exist $\varepsilon_0 > 0$ and sequences $(x_n), (y_n) \subset K$ with $|x_n - y_n| \to 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for every n.

Since K is bounded, by the Bolzano–Weierstrass theorem the sequence (x_n) has a convergent subsequence $x_{n_k} \to x^*$; since K is closed, $x^* \in K$. Moreover, since $(|x_{n_k} - y_{n_k}|)_k$ is a subsequence of $(|x_n - y_n|)_n$ and the latter tends to 0, we also have $|x_{n_k} - y_{n_k}| \to 0$; hence

$$|y_{n_k} - x^*| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - x^*| \longrightarrow 0,$$

so $y_{n_k} \to x^*$ as well. By continuity of f at x^*, $f(x_{n_k}) \to f(x^*)$ and $f(y_{n_k}) \to f(x^*)$, hence

$$|f(x_{n_k}) - f(y_{n_k})| \longrightarrow |f(x^*) - f(x^*)| = 0,$$

contradicting $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon_0$ for every k. Contradiction. ■

Corollary 2.1. *On a compact set K, continuity and uniform continuity coincide: $C(K)$ coincides with the set of uniformly continuous functions on K.*

3. Lipschitz Continuity

Definition 3.1 (Lipschitz Function)**.** *$f : I \to \mathbb{R}$ is **Lipschitz** (with constant L) if there exists $L \geq 0$ such that*

$$|f(x) - f(y)| \leq L\,|x - y| \qquad \forall\, x, y \in I.$$

Proposition 3.1. *Every Lipschitz function on I is uniformly continuous on I.*

Proof. Let L be the Lipschitz constant. If $L = 0$, f is constant and the claim is trivial. If $L > 0$, given $\varepsilon > 0$ it suffices to choose $\delta = \varepsilon/L$: then $|x - y| < \delta \Rightarrow |f(x) - f(y)| \leq L|x - y| < L\delta = \varepsilon$. ■

To construct examples of Lipschitz functions, the bounded-derivative criterion is useful, and it descends from Lagrange's theorem. It is worth clarifying right away the relationship between Rolle's and Lagrange's theorems, which is often stated imprecisely: from a *logical* standpoint, Rolle's theorem is a special case of Lagrange's – it is obtained by setting $f(a) = f(b)$ in Lagrange's conclusion, which then becomes $f'(c) = 0$. The standard proof of Lagrange's theorem, however, proceeds in the opposite direction, *using* Rolle's theorem (applied not to f but to a suitable auxiliary function that "straightens" the chord between $(a, f(a))$ and $(b, f(b))$); for this reason Rolle is traditionally stated and proved first, even though, as stated, it is less general. We recall here only the statements of these two classical results (for the proofs, which are standard, see e.g. [1, 3, 2]), since they are the direct ingredients of Corollary 3.1, which is what interests us.

Theorem 3.1 (Rolle)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), with $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Theorem 3.2 (Lagrange – Mean Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 3.1 (Bounded Derivative $\Leftrightarrow$ Lipschitz Continuity, for Differentiable Functions)**.** *Let I be an interval and $f : I \to \mathbb{R}$ differentiable on I. Then f is Lipschitz on I with constant L if and only if $|f'(x)| \leq L$ for every $x \in I$.*

Proof. ($\Leftarrow$) Let $x, y \in I$, $x < y$. Applying Lagrange's Theorem (3.2) on $[x, y] \subseteq I$, there exists $c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x)$. Then

$$|f(y) - f(x)| = |f'(c)|\,|y - x| \leq L\,|y - x|.$$

($\Rightarrow$) Suppose f is Lipschitz on I with constant L: $|f(x)-f(y)| \le L|x-y|$ for every $x, y \in I$. Fix $x \in I$; for every $h \neq 0$ such that $x+h \in I$ we have

$$\left|\frac{f(x+h)-f(x)}{h}\right| \le \frac{L\,|h|}{|h|} = L.$$

Since f is differentiable at x, the difference quotient on the left converges to $f'(x)$ as $h \to 0$; passing to the limit in an inequality (the absolute value is a continuous function) gives $|f'(x)| \le L$. As $x \in I$ was arbitrary, $|f'(x)| \le L$ for every $x \in I$. ■

Remark 3.2. *The direction* ($\Rightarrow$) *just proved shows that* there cannot exist *a function that is Lipschitz and differentiable* on all of I *with an unbounded derivative: if the derivative exists at every point of* I*, it is automatically bounded by the same Lipschitz constant. Lipschitz continuity can, however, coexist with isolated points of* non-*differentiability: for instance* $f(x) = |x|$ *is Lipschitz on* $\mathbb{R}$ *with* $L = 1$ *but is not differentiable at* 0*. The equivalence in Corollary 3.1 therefore requires* f *to be differentiable on* all *of* I*, a hypothesis to be checked case by case.*

Example 3.1 ($f(x) = \sin(x)$ Is Lipschitz on $\mathbb{R}$)**.** $f(x) = \sin(x)$ *is differentiable on* $\mathbb{R}$ *with* $f'(x) = \cos(x)$*, so* $|f'(x)| \le 1$ *for every* x*. By Corollary 3.1,* sin *is Lipschitz on* $\mathbb{R}$ *with constant* $L = 1$[a]*:*

$$|\sin x - \sin y| \le |x-y| \qquad \forall\, x, y \in \mathbb{R}.$$

This argument applies, unchanged, to every *differentiable function with bounded derivative: it is not a property specific to* sin*.*

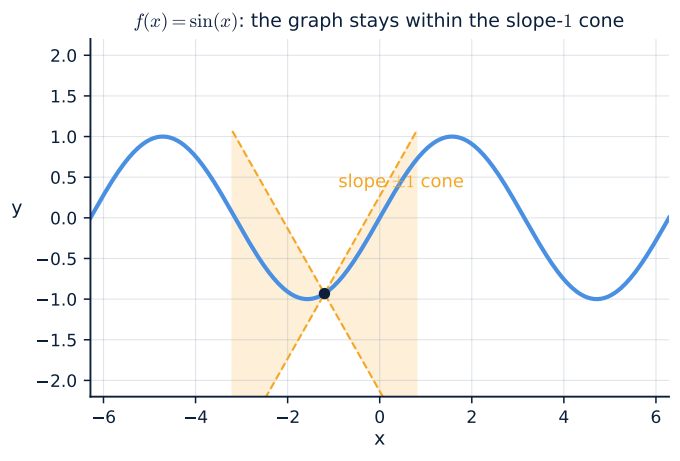


Figure 3.1. *The graph of* sin *always stays inside the slope-*± 1 *cone drawn from any point: this is the geometric translation of the Lipschitz condition with* $L = 1$*.*

[a] The inequality can also be proved directly, without going through differential calculus, using the prosthaphaeresis formula $\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$ and the elementary inequality $|\sin t| \le |t|$ for every $t \in \mathbb{R}$ (the latter follows, for instance, from the fact that sin has derivative cos bounded by 1 and $\sin 0 = 0$, i.e. it is itself a special case of Corollary 3.1): $|\sin x - \sin y| = 2\left|\cos\left(\frac{x+y}{2}\right)\right| \cdot \left|\sin\left(\frac{x-y}{2}\right)\right| \le 2 \cdot 1 \cdot \left|\frac{x-y}{2}\right| = |x-y|$.

Counterexample 3.1 ($f(x) = \sqrt{x}$ Is Not Lipschitz on $[0,1]$)**.** *If f were Lipschitz on $[0,1]$, there would exist L such that $|\sqrt{x} - \sqrt{0}| \le L|x - 0|$ for every $x, y \in [0,1]$; in particular, taking $y = 0$ and restricting to $x \in (0,1]$ (the only restriction needed is $x \neq 0$, to be able to divide by x in the next step), we would get $\sqrt{x} \le Lx$, i.e. $1/\sqrt{x} \le L$ for every $x \in (0,1]$. But $1/\sqrt{x} \to +\infty$ as $x \to 0^+$, so no finite L can satisfy the inequality for every x: a contradiction. The geometric phenomenon is that the slope of the graph of $\sqrt{x}$ at 0 is infinite (a vertical tangent line).*

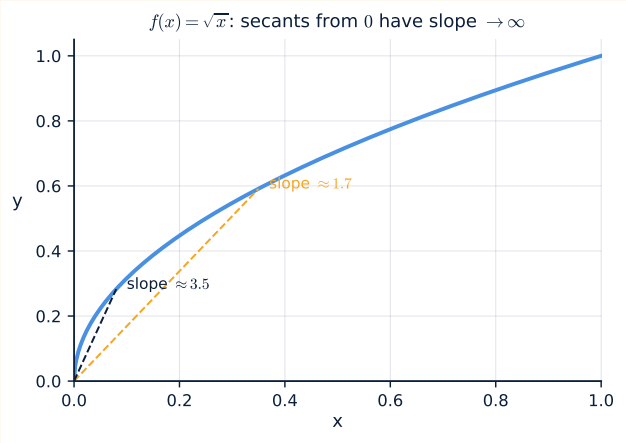


Figure 3.2. *The secants from 0 to points closer and closer to the origin have ever-larger slope: no constant L can bound them all.*

Subtle Case 3.1 (Local vs. Global Lipschitz Continuity)**.** *A common source of confusion is the distinction between* local *Lipschitz continuity (on every bounded subinterval) and* global *Lipschitz continuity (on the whole domain). For example $f(x) = x^2$ is Lipschitz on every bounded interval $[a,b]$ (by Corollary 3.1, with $L = \sup_{[a,b]} |2x| = 2\max(|a|,|b|)$), but it is* not *Lipschitz on all of $\mathbb{R}$: if it were, with constant L, taking $x = L + 1$, $y = 0$ would give $|f(x) - f(y)| = (L+1)^2 > L(L+1) = L|x - y|$, a contradiction. The Lipschitz constant, when it exists, generally depends on the size of the interval considered.*

4. Hölder Continuity

Definition 4.1 (Hölder Function)**.** *Let $\alpha \in (0,1]$. $f : I \to \mathbb{R}$ is **Hölder continuous of exponent** α (written $f \in C^{0,\alpha}(I)$) if there exists $C \geq 0$ such that*

$$|f(x) - f(y)| \leq C\,|x-y|^{\alpha} \qquad \forall\, x, y \in I.$$

Proposition 4.1 (Hölder Continuity ⇒ Uniform Continuity)**.** *If $f : I \to \mathbb{R}$ is Hölder of exponent α and constant C (on any interval I, even unbounded), then f is uniformly continuous on I.*

Proof. If $C = 0$, f is constant and the claim is trivial. If $C > 0$, given $\varepsilon > 0$ it suffices to choose $\delta = (\varepsilon/C)^{1/\alpha}$: then $|x-y| < \delta \Rightarrow |f(x) - f(y)| \leq C|x-y|^{\alpha} < C\delta^{\alpha} = \varepsilon$. Note that, unlike Theorem 4.1, no boundedness hypothesis on I is needed here: the estimate is direct. ■

Remark 4.2. *For $\alpha = 1$ we recover exactly the definition of Lipschitz continuity: $C^{0,1}(I)$ is the set of Lipschitz functions on I. By Proposition 4.1, every Hölder function (for any α) is already uniformly continuous: the exponent α therefore measures an intermediate degree of regularity between uniform continuity and Lipschitz continuity – the closer α is to 1, the more stringent the condition.*

Theorem 4.1 (Hierarchy of Hölder Classes on Bounded Domains)**.** *Let I be a bounded interval (of diameter $D := \sup_{x,y\in I} |x-y| < \infty$) and let $0 < \alpha_1 \leq \alpha_2 \leq 1$. Then*

$$C^{0,\alpha_2}(I) \subseteq C^{0,\alpha_1}(I).$$

Proof. Let $f \in C^{0,\alpha_2}(I)$, with $|f(x) - f(y)| \leq C|x-y|^{\alpha_2}$ for every $x, y \in I$. For $x, y \in I$ with $x \neq y$ write, using $|x-y| \leq D$ and $\alpha_2 - \alpha_1 \geq 0$,

$$|x-y|^{\alpha_2} = |x-y|^{\alpha_1} \cdot |x-y|^{\alpha_2-\alpha_1} \leq |x-y|^{\alpha_1} \cdot D^{\alpha_2-\alpha_1},$$

where the last inequality follows because $t \mapsto t^{\alpha_2-\alpha_1}$ is increasing on $[0, +\infty)$ and $|x-y| \leq D$. Hence

$$|f(x) - f(y)| \leq C\,D^{\alpha_2-\alpha_1}\,|x-y|^{\alpha_1} =: C'\,|x-y|^{\alpha_1},$$

i.e. $f \in C^{0,\alpha_1}(I)$. ■

Remark 4.3. *The hypothesis that I is bounded is essential: on an unbounded domain the inequality $|x-y|^{\alpha_2-\alpha_1} \leq D^{\alpha_2-\alpha_1}$ makes no sense ($D = \infty$), and indeed the inclusion can fail on $\mathbb{R}$.*

Example 4.1 ($f(x) = \sqrt{x}$ Is $\frac{1}{2}$-Hölder but Not Lipschitz)**.** *On $[0,1]$, $f(x) = \sqrt{x}$ satisfies*

$$|\sqrt{x} - \sqrt{y}| \leq |x-y|^{1/2} \qquad \forall\, x, y \in [0,1],$$

so $f \in C^{0,1/2}([0,1])$ (the inequality, valid more generally for every $x, y \geq 0$, was already

proved in Example 2.1, where it was used for the uniform continuity of $\sqrt{x}$: there we showed $|\sqrt{x}-\sqrt{y}| \le \sqrt{|x-y|} = |x-y|^{1/2}$). By Counterexample 3.1, $\sqrt{x}$ is not *Lipschitz on $[0,1]$: it is a genuine example of a function that is Hölder but not Lipschitz, sitting exactly midway in the hierarchy (see the graph in Counterexample 3.1: the same vertical tangent at 0 that rules out Lipschitz continuity is compatible with the exponent $\frac{1}{2}$).*

Counterexample 4.1 (A Uniformly Continuous Function That Is Not Hölder for Any Exponent)**.** *Define $f : [0, \frac{1}{2}] \to \mathbb{R}$ by*

$$f(x) = \begin{cases} \dfrac{1}{\ln(1/x)} & 0 < x \le \frac{1}{2} \\ 0 & x = 0. \end{cases}$$

f is continuous on $[0, \frac{1}{2}]$*: it is continuous on $(0, \frac{1}{2}]$ (a composition of continuous functions with $\ln(1/x) \neq 0$ for $x \le 1/2$), and $\lim_{x\to 0^+} \frac{1}{\ln(1/x)} = 0 = f(0)$ since $\ln(1/x) \to +\infty$. Since $[0, \frac{1}{2}]$ is compact, by the Heine–Cantor theorem (2.1) f is automatically* uniformly continuous*.*

f is not Hölder for any $\alpha \in (0, 1]$*: if it were, with exponent α and constant C, taking $y = 0$ would give, for every $x \in (0, \frac{1}{2}]$,*

$$\frac{1}{\ln(1/x)} = |f(x) - f(0)| \le C\, x^{\alpha}, \qquad \textit{i.e.} \qquad x^{\alpha} \ln\frac{1}{x} \ge \frac{1}{C} \quad \forall\, x \in (0, \tfrac{1}{2}].$$

But with the substitution $t = \ln(1/x) \to +\infty$ (i.e. $x = e^{-t}$),

$$x^{\alpha} \ln\frac{1}{x} = e^{-\alpha t}\, t \longrightarrow 0 \qquad (t \to +\infty),$$

because the exponential $e^{-\alpha t}$ (with $\alpha > 0$) decays faster than t grows. So $x^{\alpha} \ln(1/x) \to 0$ as $x \to 0^+$, contradicting $x^{\alpha} \ln(1/x) \ge 1/C > 0$ for every x. No constant C can work, for any $\alpha \in (0, 1]$: f does not belong to $C^{0,\alpha}([0, \frac{1}{2}])$ for any α.

This counterexample shows that the inclusion $C^{0,\alpha}(I) \subsetneq C(I)$ is always *strict, even on compact domains where uniform continuity and continuity coincide: there exist uniformly continuous functions that are "borderline", with a logarithmic modulus of continuity ($\omega(t) \sim 1/\ln(1/t)$) too weak to fit into any Hölder class.*

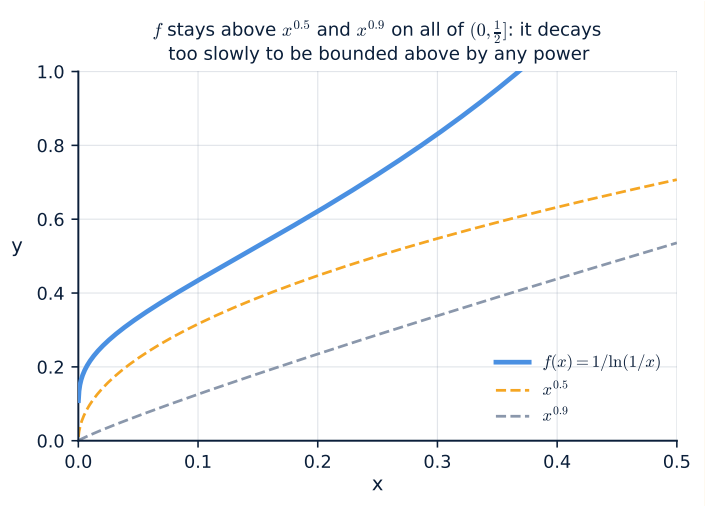


Figure 4.1. *The graph of $f(x) = 1/\ln(1/x)$ always stays above the powers $x^{0.5}$ and $x^{0.9}$ on all of $(0, \frac{1}{2}]$: it decays too slowly to be bounded above by any power, which makes Hölder continuity impossible for any α.*

The Cantor function, which we will revisit in full detail in Subtle Case 5.1 of the next section, is already an early example of a Hölder function that behaves surprisingly with respect to absolute continuity.

5. Absolute Continuity

The previous sections compared continuity, uniform continuity, Lipschitz continuity and Hölder continuity along a single chain of implications. We introduce here a fifth notion – **absolute continuity** – which deserves its own treatment: it is not simply another link in the same chain, but another *direction* of regularity, tied not to the pointwise modulus of continuity but to the behavior of the function on families of intervals. It is precisely the condition that makes the *fundamental theorem of integral calculus* hold in its most general form (Lebesgue integration): for an absolutely continuous function, and only for such a function, the integral of the derivative reconstructs the original function.

Definition 5.1 (Absolute Continuity)**.** $f : [a,b] \to \mathbb{R}$ *is* ***absolutely continuous*** *(written* $f \in AC([a,b])$*) if*

$$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 \;:\; \forall\, n \in \mathbb{N}, \;\; \forall\, (x_i, y_i)_{i=1}^{n} \subset [a,b] \text{ disjoint},$$

$$\sum_{i=1}^{n} |y_i - x_i| < \delta \quad \Longrightarrow \quad \sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon.$$

Remark 5.2. *Taking* $n = 1$ *in Definition 5.1 recovers exactly the definition of uniform continuity: absolute continuity is therefore a condition* at least *as strong as uniform continuity, because it must hold not on a single interval but* simultaneously *on an arbitrary (finite) number of disjoint intervals, provided the sum of their lengths is small.*

Proposition 5.1 (Lipschitz Continuity ⇒ Absolute Continuity)**.** *If* $f : [a,b] \to \mathbb{R}$ *is Lipschitz with constant* L*, then* $f \in AC([a,b])$*.*

Proof. Given $\varepsilon > 0$ set $\delta := \varepsilon / L$ (if $L = 0$, f is constant and there is nothing to prove). For every finite family of disjoint intervals $(x_i, y_i) \subset [a,b]$, $i = 1, \dots, n$, with $\sum_i |y_i - x_i| < \delta$, we have

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| \le \sum_{i=1}^{n} L\, |y_i - x_i| = L \sum_{i=1}^{n} |y_i - x_i| < L\delta = \varepsilon.$$

■

Proposition 5.2 (Absolute Continuity ⇒ Uniform Continuity)**.** *If* $f \in AC([a,b])$*, then* f *is uniformly continuous on* $[a,b]$*.*

Proof. Immediate from Definition 5.1 applied with $n = 1$: given $\varepsilon > 0$ let $\delta > 0$ be the one provided by absolute continuity; then for any single pair $x, y \in [a,b]$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$, which is exactly the definition of uniform continuity. ■

Theorem 5.1 (Fundamental Theorem of Calculus, Lebesgue Version – Stated Without Proof)**.** *Let* $g \in L^1([a,b])$[a] *and let* $F(x) := \displaystyle\int_a^x g(t)\, dt$*. Then* $F \in AC([a,b])$ *and* $F' = g$

almost everywhere. Conversely, if $F \in AC([a,b])$, then F is differentiable almost everywhere, $F' \in L^1([a,b])$, and

$$F(b) - F(a) = \int_a^b F'(t)\,dt.$$

[a]Recall that $L^1([a,b])$ denotes the space of functions $g : [a,b] \to \mathbb{R}$ that are (Lebesgue) measurable and summable, i.e. such that $\int_a^b |g(t)|\,dt < \infty$; the integral is the Lebesgue integral, which extends the Riemann integral and allows a broader class of functions to be integrated (including discontinuous functions, as long as they are measurable).

Proof. A classical result from measure theory, which requires the construction of the Lebesgue integral and the Lebesgue differentiation theorem: we do not prove it here and refer to [6]. We state it because it is precisely this theorem that makes absolute continuity the "right" notion: it is the minimal condition under which $F(b) - F(a) = \int_a^b F'$ holds. ■

Example 5.1 (An Absolutely Continuous Function That Is Not Hölder for Any Exponent)**.** *Let us return to the function from Counterexample 4.1, $f(x) = 1/\ln(1/x)$ on $(0, \frac{1}{2}]$, $f(0) = 0$, already shown not to be Hölder for any α. We show that it is nonetheless absolutely continuous, by exhibiting it as the integral of an L^1 function. Set*

$$g(t) := \frac{1}{t\,\ln^2(1/t)}, \qquad t \in (0, \tfrac{1}{2}].$$

With the substitution $u = \ln(1/t)$ (so $t = e^{-u}$, $dt = -e^{-u}\,du$, and $t\ln^2(1/t) = e^{-u}u^2$) we get

$$\int_x^{1/2} g(t)\,dt = \int_x^{1/2} \frac{dt}{t\ln^2(1/t)} = \int_{\ln 2}^{\ln(1/x)} \frac{du}{u^2} = \frac{1}{\ln 2} - \frac{1}{\ln(1/x)},$$

a quantity that stays bounded (and converges to $1/\ln 2$) as $x \to 0^+$: so $g \in L^1((0, \frac{1}{2}])$. The same computation shows that

$$F(x) := \int_0^x g(t)\,dt = \frac{1}{\ln 2} - \left(\frac{1}{\ln 2} - \frac{1}{\ln(1/x)}\right) = \frac{1}{\ln(1/x)} = f(x)$$

(up to the irrelevant additive constant, absorbed by setting $F(0) = 0$). By Theorem 5.1, $F = f$ is absolutely continuous on $[0, \frac{1}{2}]$, despite not being Hölder for any exponent. This shows that

$$AC([a,b]) \not\subseteq \bigcup_{\alpha \in (0,1]} C^{0,\alpha}([a,b]) :$$

absolute continuity does not imply *Hölder continuity.*

Remark 5.3 (Hölder Continuity and Absolute Continuity Are Independent)**.** *Combining Example 5.1 with Subtle Case 5.1 yields an important fact, often left unsaid in introductory courses:* ***neither notion implies the other.***

- *The Cantor function is Hölder (with $\alpha = \log_3 2$), hence in particular uniformly continuous, but is* not *absolutely continuous: $C^{0,\alpha}([a,b]) \not\subseteq AC([a,b])$.*
- *The function $f(x) = 1/\ln(1/x)$ from Example 5.1 is absolutely continuous but is* not *Hölder for any α: $AC([a,b]) \not\subseteq C^{0,\alpha}([a,b])$.*

Both classes, however, are strictly contained in the set of uniformly continuous functions

(Proposition 5.2 and Theorem 4.1), and both contain the Lipschitz functions (Proposition 5.1). The hierarchy of regularity, updated with this new information, is therefore not a single chain but forks into two parallel branches right after Lipschitz continuity: see Section 6.

Remark 5.4 (The Intersection Is Larger Than Lipschitz Continuity Alone)**.** *The Lipschitz functions are contained in* both *classes $C^{0,\alpha}([a,b])$ and $AC([a,b])$ (Theorem 4.1 with $\alpha_1 = \alpha, \alpha_2 = 1$, and Proposition 5.1), so they belong to their intersection. But the intersection is* strictly larger*: the function $f(x) = \sqrt{x}$ on $[0,1]$ is both $\frac{1}{2}$-Hölder (Example 4.1) and absolutely continuous – indeed $f(x) = \int_0^x \frac{1}{2\sqrt{t}}\,dt$, and $g(t) = \frac{1}{2\sqrt{t}}$ is integrable on $[0,1]$ because $\int_0^1 g(t)\,dt = \left[\sqrt{t}\right]_0^1 = 1 < \infty$, so $g \in L^1([0,1])$ and by Theorem 5.1 $f \in AC([0,1])$ – yet f is* not *Lipschitz (Counterexample 3.1). Hence*

$$\mathit{Lipschitz}([a,b]) \subsetneq C^{0,\alpha}([a,b]) \cap AC([a,b]),$$

a strict inclusion: in the diagram in Section 6, the Lipschitz functions occupy only part of the overlap zone between Hölder and absolutely continuous functions, not all of it.

Subtle Case 5.1 (The Cantor Function: Construction, Hölder Continuity, and Why It Is Not Absolutely Continuous)**.** *The* Cantor function *(or "devil's staircase") is one of the most instructive – and at the same time most treacherous – examples in the whole of real analysis: it rarely appears in secondary school, and is often only briefly mentioned in first-year university courses. It is worth constructing it carefully.*

Construction of the Cantor set. *Start with $C_0 = [0,1]$. To obtain C_1, remove from C_0 the open middle third $(\frac{1}{3}, \frac{2}{3})$, giving $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Repeat the procedure on each of the two remaining segments (removing the middle third of each), giving C_2 (four segments of length $\frac{1}{9}$), and so on: C_{n+1} is obtained from C_n by removing the open middle third of each of the 2^n segments of C_n (each of length 3^{-n}). The* ***Cantor set*** *is $\mathcal{C} := \bigcap_{n\geq 0} C_n$: a closed, uncountable set of Lebesgue measure zero (the total length removed is $\sum_{n\geq 0} 2^n \cdot 3^{-(n+1)} = \frac{1}{3}\sum_{n\geq 0}(\frac{2}{3})^n = \frac{1}{3}\cdot 3 = 1$, i.e. "all" of the length is removed, and yet $\mathcal{C}$ is not empty: it contains, for instance, all the segment endpoints, and much more).*

Construction of the Cantor function. *Define $\varphi_0(x) := x$ on $[0,1]$, and construct φ_{n+1} recursively from φ_n: on each middle third removed at step $n+1$, "flatten" φ_n to the average value it takes at the two endpoints of that third, and linearly rescale φ_n on the two remaining segments so that it joins up continuously. For example φ_1 equals $\frac{1}{2}$ on $[\frac{1}{3}, \frac{2}{3}]$ and is linear (from 0 to $\frac{1}{2}$, then from $\frac{1}{2}$ to 1) on the two remaining segments. One shows that the sequence (φ_n) converges uniformly to a function $\varphi : [0,1] \to [0,1]$ – the* ***Cantor function*** *– continuous, monotone nondecreasing, with $\varphi(0) = 0$, $\varphi(1) = 1$, and constant on every removed interval (e.g. $\varphi \equiv \frac{1}{2}$ on $(\frac{1}{3}, \frac{2}{3})$, $\varphi \equiv \frac{1}{4}$ on $(\frac{1}{9}, \frac{2}{9})$, $\varphi \equiv \frac{3}{4}$ on $(\frac{7}{9}, \frac{8}{9})$, and so on).*

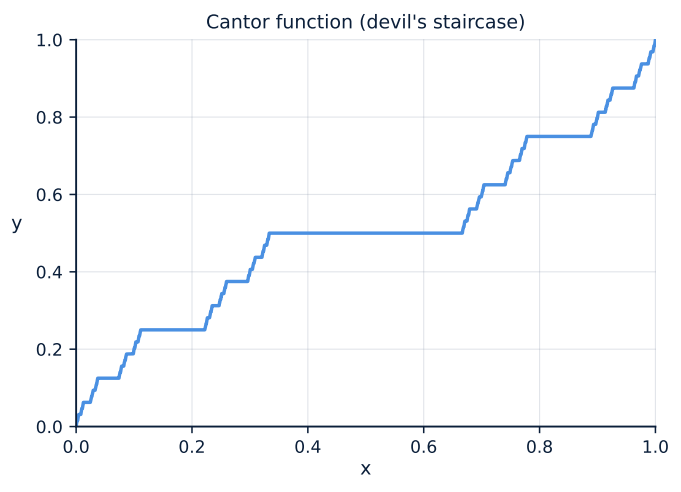


Figure 5.1. *The Cantor function φ on $[0,1]$: constant on every removed interval, yet it rises from 0 to 1.*

Why it is Hölder with $\alpha = \log_3 2$. *At each step of the construction, an interval of length 3^{-n} (a segment of C_n) is mapped by φ to an interval of length 2^{-n} (the variation of φ halves at each step, while the length is divided by 3). For x, y in the same segment of C_n of length 3^{-n} we therefore have $|\varphi(x) - \varphi(y)| \leq 2^{-n}$. We look for α such that this estimate can be written as $C|x-y|^\alpha$ with $|x-y| \leq 3^{-n}$: we want $2^{-n} = (3^{-n})^\alpha$, i.e. $2^{-n} = 3^{-n\alpha}$, i.e. (taking logarithms) $-n \ln 2 = -n\alpha \ln 3$, hence*

$$\alpha = \frac{\ln 2}{\ln 3} \approx 0.6309.$$

With some care for the case where x, y fall in different segments (using the monotonicity of φ and summing the estimates over the intermediate segments), one obtains $|\varphi(x) - \varphi(y)| \leq C|x-y|^{\ln 2/\ln 3}$ for every $x, y \in [0,1]$, with C an absolute constant: $\varphi \in C^{0,\log_3 2}([0,1])$.

Why it is not absolutely continuous. *Outside the Cantor set $\mathcal{C}$ (i.e. on the infinitely many removed intervals), φ is locally constant, hence $\varphi' = 0$. Since $\mathcal{C}$ has Lebesgue measure zero, we conclude that $\varphi' = 0$* almost everywhere *on $[0,1]$. If φ were absolutely continuous, Theorem 5.1 would give*

$$\varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)\, dt = \int_0^1 0\, dt = 0,$$

but $\varphi(1) - \varphi(0) = 1 - 0 = 1 \neq 0$: a contradiction. Hence $\varphi \notin AC([0,1])$, despite being continuous, monotone, and even Hölder.

This is the heart of the "paradox" of the Cantor function: all of the growth of φ (from 0 to 1) happens on a set of points of measure zero, where we cannot "see" the derivative in order to reconstruct the function by integrating it. It is the standard example, in second-year analysis or measure theory courses, showing that continuity, monotonicity, and even Hölder continuity are not enough to guarantee that the integral of the derivative reconstructs the function: only absolute continuity guarantees it.

6. Synthesis: The Complete Hierarchy

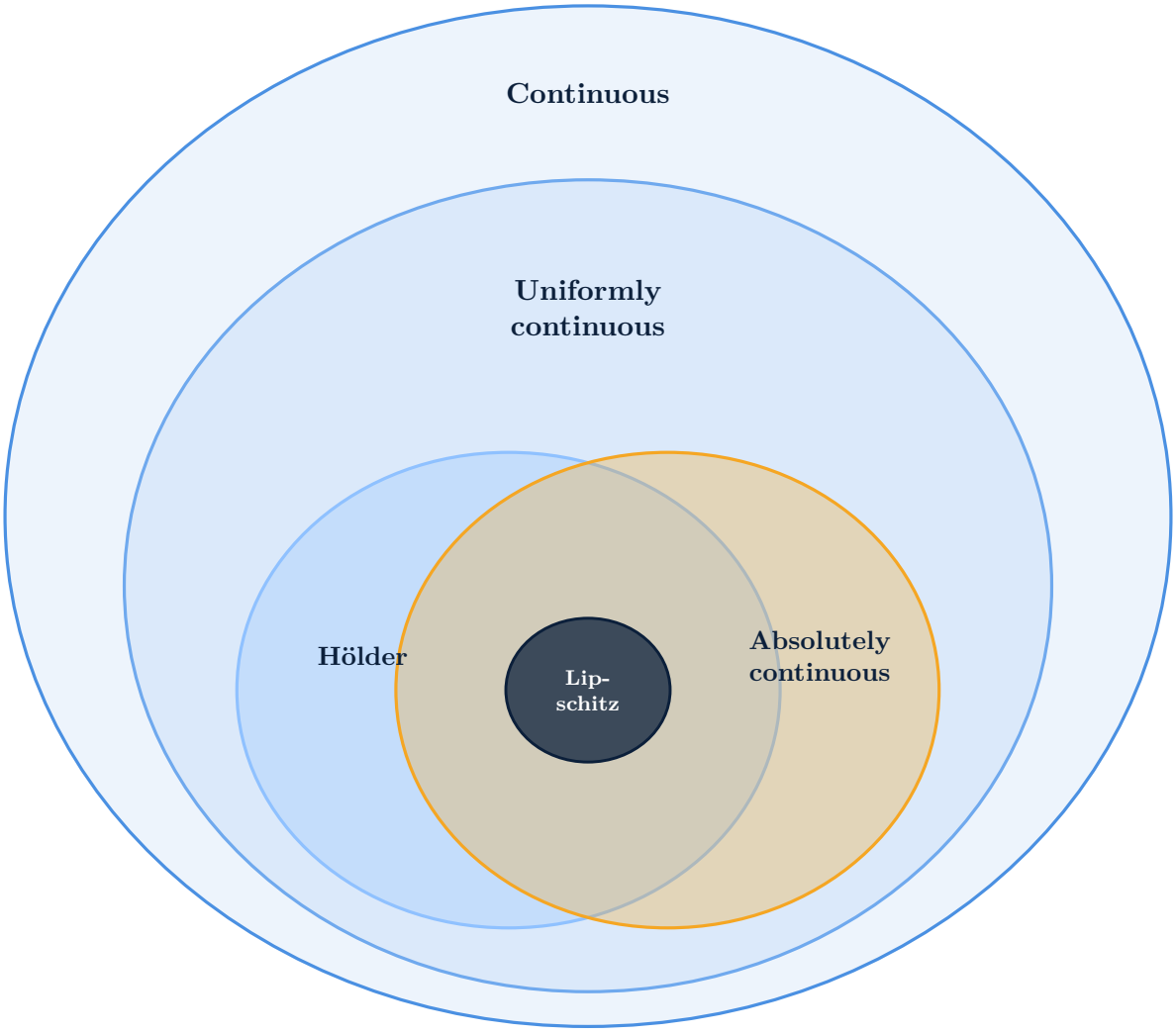
Theorem 6.1 (Hierarchy of Regularity)**.** *Let $I = [a, b] \subseteq \mathbb{R}$ be a closed, bounded interval. Then, for every $\alpha \in (0, 1)$,*

$$\textit{Lipschitz}(I) \subseteq C^{0,\alpha}(I) \subseteq \textit{Uniformly continuous}(I) = C(I),$$

$$\textit{Lipschitz}(I) \subseteq AC(I) \subseteq \textit{Uniformly continuous}(I) = C(I),$$

where the equality on the right is the Heine–Cantor theorem (2.1). All the inclusions shown are strict in general (Counterexamples 3.1 and 4.1), and, more subtly, ***the classes*** $C^{0,\alpha}(I)$ ***and*** $AC(I)$ ***are not comparable****: neither contains the other (Remark 5.3). The hierarchy, therefore, is not a single chain but forks into two parallel branches right after Lipschitz continuity, and then reconverges at uniform continuity.*

The figure below illustrates the hierarchy just proved. The "Hölder" and "Absolutely continuous" classes are drawn as two sets that partially overlap (neither contains the other), both contained in the uniformly continuous functions; the Lipschitz functions form a smaller circle, drawn explicitly *inside* their overlap zone but without exhausting it (Remark 5.4).



The zone where the two central circles overlap represents the functions that are both Hölder and absolutely continuous; the small dark circle inside it is the set of Lipschitz functions, which always sits inside it (Proposition 5.1 and Theorem 4.1) but does not fill it: $f(x) = \sqrt{x}$ (Remark 5.4) belongs to the overlap without being Lipschitz.

Summary Table

Notion	**Definition (in short)**	**Example**	**Counterexample**	**Subtle case**
Continuity (pointwise)	$\delta = \delta(\varepsilon, x_0)$	$1/x$ on $(0,1)$	step at 0	Dirichlet / Thomae
Uniform continuity	$\delta = \delta(\varepsilon)$, uniform in x, y	$\sqrt{x}$ on $[0,\infty)$	$1/x$ on $(0,1)$	x^2 on $\mathbb{R}$
Hölder continuity (α)	$\lvert f(x)-f(y)\rvert \le C\lvert x-y\rvert^{\alpha}$	$\sqrt{x}$ (α=$\frac{1}{2}$)	x^2 on $\mathbb{R}$ (not Hölder for any α on unbounded domain)	Cantor function ($\alpha = \log_3 2$)
Absolute continuity	δ on finite sums of disjoint intervals	$\frac{1}{\ln(1/x)}$ on $[0,\frac{1}{2}]$ (not Hölder, Ex. 5.1)	Cantor function (not AC)	AC and Hölder continuity are independent
Lipschitz continuity	$\lvert f(x)-f(y)\rvert \le L\lvert x-y\rvert$	$\sin x$ on $\mathbb{R}$ (L=1)	$\sqrt{x}$ on $[0,1]$	local vs. global (x^2)

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