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A Problem on the Ellipse, Step by Step

A guided solution to a problem with a parameter, a parabola and a family of lines

High School Mathematics Study Notes

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The problem

> *After determining for which values of a the equation*
>
> $$(a-3)x^2 + (a-6)y^2 = a - 3$$
>
> *represents an ellipse with foci on the y-axis, consider the ellipse passing through the point $\left(\frac{1}{2}, \sqrt{3}\right)$. Let A, B, A_1, B_1 be its vertices (named counterclockwise, with A having positive x-coordinate). Determine:*
>
> ***a.*** *the area of the square inscribed in the ellipse with sides parallel to the Cartesian axes;*
>
> ***b.*** *the equation of the parabola with vertex at A and passing through B and B_1;*
>
> ***c.*** *for which values of $k \neq 0$ the lines of the family $4kx + ky = k - 4$ meet the arc of the parabola lying in the first quadrant.*

Source: M. Bergamini, G. Barozzi, A. Trifone, *Matematica.blu 2.0*, vol. 3, 4th edition, Zanichelli — Chapter 7 (Ellipse), "Minisimulazione di Matematica" (mock-exam problem set).

The strategy in brief

The problem looks long, but it has three levels:

1. **Understand the equation.** For which a is it an ellipse with foci on the y-axis? (Step 0)
2. **Pin down the ellipse.** The given point tells us the value of a. From there we obtain the equation and the vertices. (Step 1)
3. **Answer the three questions.** Each one uses the ellipse and its vertices, and question c also uses the parabola found in b. (Parts a, b, c)

Each step has a guiding idea, worth keeping in mind before any computation:

- **Step 0:** observe the coefficients *before* manipulating the equation.
- **Step 1:** require the curve to pass through the given point.
- **Part a:** exploit the symmetry of the ellipse.
- **Part b:** choose the type of parabola by looking at the given points.

- **Part c:** slide the lines of the family along the arc and look for the limiting positions.

Each step feeds the next: a mistake in the ellipse carries over into every question. That is why, at the end of each section, there is a short recap of what we know.

Step 0 — When is the equation an ellipse with foci on the y-axis?

Where we are. We have the equation

$$(a-3)x^2 + (a-6)y^2 = a-3,$$

in which the letter a is a **parameter**: a fixed real number, not yet known, that changes the curve depending on the value chosen. For some values of a the curve will be an ellipse, for others it will not.

Goal. Find all the values of a for which the equation represents an ellipse with foci on the y-axis.

Idea. Bring the equation to canonical form (Memento M1, below) and read its denominators. Before manipulating anything, though, let us observe the **coefficients**: where they vanish, the equation changes structure and must be treated separately.

Memento M1 — The ellipse in canonical form

An equation of the type

$$\frac{x^2}{p^2} + \frac{y^2}{s^2} = 1, \qquad p > 0,\ s > 0$$

represents an ellipse centered at the origin, symmetric with respect to both axes. The numbers p and s are the lengths of the **semi-axes**. The ellipse crosses the x-axis at the points $(\pm p, 0)$ and the y-axis at the points $(0, \pm s)$. These four points are the **vertices**.
The **foci** lie on the axis along which the ellipse is more elongated, that is, the axis with the larger semi-axis:

- on the y-axis if $s > p$;
- on the x-axis if $p > s$.

For positive numbers, $s > p \iff s^2 > p^2$ (the symbol $\iff$ reads "if and only if": the two statements are equivalent, each holds exactly when the other does). In the equation, p^2 and s^2 are precisely the denominators. Therefore:
foci on the y-axis $\iff$ the denominator of y^2 is greater than the denominator of x^2.

(In textbooks the semi-axes are called a and b. Here we use p and s because a is already the parameter of the problem.)

0.1 — The coefficients. In the canonical form of M1 there is a term with x^2, a term with y^2 and a 1 on the right-hand side. Let us observe the numbers multiplying x^2 and y^2:

$$\underbrace{(a-3)}_{\text{coefficient of } x^2}\, x^2 + \underbrace{(a-6)}_{\text{coefficient of } y^2}\, y^2 = a - 3.$$

Each of them vanishes for one value of a: the first for $a = 3$, the second for $a = 6$. In those cases one of the two quadratic terms **disappears** and the equation no longer has the structure of M1. What is left?

0.2 — The two special cases. In both, the ellipse is said to **degenerate** into one or two lines.

Case $a = 3$. We substitute directly into the equation:

$$0 \cdot x^2 + (3-6)y^2 = 0 \implies -3y^2 = 0 \implies y^2 = 0 \implies y = 0.$$

(The symbol $\implies$ reads "implies": if what is on the left holds, then what is on the right holds.) Here $y = 0$ is the equation of the x-axis, that is, a **line**, not an ellipse.

Case $a = 6$. We substitute directly:

$$(6-3)x^2 + 0 \cdot y^2 = 6 - 3 \implies 3x^2 = 3 \implies x^2 = 1 \implies x = \pm 1.$$

We get the **two vertical lines** $x = 1$ and $x = -1$: again, not an ellipse.

From now on we can therefore assume $a \neq 3$ and $a \neq 6$.

0.3 — The general case: $a \neq 3$ and $a \neq 6$. Now we can divide both sides by $a - 3$ (allowed, since it is not zero):

$$x^2 + \frac{a-6}{a-3}\,y^2 = 1. \qquad (*)$$

In canonical form the y^2 term must have a **denominator**. Its coefficient is the fraction $\frac{a-6}{a-3}$, and dividing by a fraction is the same as multiplying by its reciprocal. So we can write

$$\frac{a-6}{a-3}\,y^2 = \frac{y^2}{\frac{a-3}{a-6}},$$

and the equation becomes

$$\frac{x^2}{1} + \frac{y^2}{\frac{a-3}{a-6}} = 1,$$

from which we read the denominators:

$$p^2 = 1, \qquad s^2 = \frac{a-3}{a-6}.$$

Observation on the method. The two special cases must be treated separately for a reason that concerns the *equation*: for $a = 3$ and for $a = 6$ a quadratic term disappears and the curve reduces to lines (0.2).
There is also a second point, **different from the first**, which concerns the *method* we are using. In the steps we just took we had to divide by $a - 3$ and then write the reciprocal $\frac{a-3}{a-6}$, which exists only if $a \neq 6$. For $a = 3$ and $a = 6$ these operations could not be carried out (they would be divisions by zero): so the method would not cover those two values anyway.
Let us note the difference: the values $a = 3$ and $a = 6$ are **allowed by the given equation** (they are admissible values of the parameter), but they are **not compatible with the method** we chose. A solving method does not always cover every case of the given problem: before applying it, it is worth asking for which values it works, and treating the others separately, as we have done.

0.4 — The condition on the foci. By M1 the foci are on the y-axis if the denominator of y^2 exceeds that of x^2:

$$\frac{a-3}{a-6} > 1.$$

If this condition holds, s^2 is greater than 1, hence positive. So both denominators are positive and the equation really is that of an ellipse.

To solve the inequality we move everything to the left and put it over a common denominator:

$$\begin{aligned} \frac{a-3}{a-6} - 1 > 0 &\implies \frac{(a-3)-(a-6)}{a-6} > 0 \\ &\implies \frac{a-3-a+6}{a-6} > 0 \\ &\implies \frac{3}{a-6} > 0. \end{aligned}$$

A fraction with positive numerator (3) is positive only if its denominator is positive:

$$a-6 > 0 \implies a > 6.$$

Watch out — It is better not to multiply by a quantity of unknown sign

Faced with $\frac{a-3}{a-6} > 1$, one is tempted to "clear the denominator" by multiplying by $a-6$. It is not forbidden, but it is inconvenient. Multiplying an inequality by a positive number preserves its direction, while multiplying by a negative number reverses it. Here we do not know whether $a-6$ is positive or negative, so we would have to split into two cases and solve them one at a time.
The more convenient method is to **move everything to one side, compare with zero and reduce to a single fraction**. The sign of a fraction is studied by looking at numerator and denominator, with no risk of getting the direction wrong.

Result of Step 0. The equation represents an ellipse with foci on the y-axis if and only if

$$\boxed{a > 6}.$$

This range is already compatible with $a \neq 3$ and $a \neq 6$.

Let's recap — Step 0

- For $a = 3$ and $a = 6$ the equation is not an ellipse: it reduces to lines (the ellipse **degenerates**).
- For all other a, where the method (division by $a - 3$ and reciprocal) is applicable, the equation is equivalent to $\frac{x^2}{1} + \frac{y^2}{s^2} = 1$ with $s^2 = \frac{a-3}{a-6}$.
- The foci are on the y-axis if $s^2 > 1$, that is, if $a > 6$.

Fun fact — One equation, many characters

We have only studied $a > 6$, but with what we have already computed we can understand the other values too (except 3 and 6, already seen). After dividing by $a - 3$ the equation is $(*)$, so everything depends on the sign of the coefficient $\frac{a-6}{a-3}$ of y^2.

- $a < 3$ **or** $a > 6$: numerator and denominator have the same sign, the coefficient is positive and the curve is an ellipse. For $a > 6$ the foci are on the y-axis (Step 0). For $a < 3$ we have $s^2 - 1 = \frac{3}{a-6} < 0$, that is, $s^2 < p^2$: foci on the x-axis. Example: $a = 0$ gives $x^2 + 2y^2 = 1$.
- $3 < a < 6$: the coefficient is negative, so there is a minus sign between the two terms, $x^2 - (\dots)\, y^2 = 1$. This is the equation of another conic, the **hyperbola**. Example: $a = 4$ gives $x^2 - 2y^2 = 1$.

value of a	what it represents
$a < 3$	ellipse with foci on the x-axis
$a = 3$	a line (the x-axis)
$3 < a < 6$	hyperbola
$a = 6$	two parallel lines, $x = \pm 1$
$a > 6$	ellipse with foci on the y-axis

A curiosity: the curve is **never** a circle. That would require $\frac{a-6}{a-3} = 1$, that is, $-6 = -3$, which is impossible.

Step 1 — Finding a, the equation and the vertices

Where we are. In Step 0 we saw that the equation is an ellipse with foci on the y-axis for every $a > 6$: infinitely many ellipses, one for each value of a.

Goal. Find the value of a that corresponds to the ellipse of the problem, then write its equation and its vertices.

Idea. The text says that the ellipse passes through the point $\left(\frac{1}{2}, \sqrt{3}\right)$. By M2 (Memento below), the coordinates of the point must satisfy the equation: substituting them gives an equation in the single unknown a.

Memento M2 — When a point belongs to a curve

A point belongs to a curve if and only if its coordinates satisfy the equation of the curve. We replace x with the x-coordinate and y with the y-coordinate and check whether the equality is true.
Example with the line $r : \ y = 2x + 1$:

- $P(0,1)$: $2 \cdot 0 + 1 = 1$, equal to the y-coordinate, so $P \in r$.
- $Q(4,3)$: $2 \cdot 4 + 1 = 9 \neq 3$, so $Q \notin r$.

1.1 — We impose the passage through the point. We substitute $x = \frac{1}{2}$ and $y = \sqrt{3}$ into the equation of the problem:

$$(a-3)\left(\tfrac{1}{2}\right)^2 + (a-6)\left(\sqrt{3}\right)^2 = a - 3.$$

1.2 — We solve for a. We work out the squares and multiply both sides by 4 to get rid of the fraction:

$$(a-3) + 12(a-6) = 4(a-3) \implies 13a - 75 = 4a - 12 \implies 9a = 63 \implies a = 7.$$

1.3 — Check. A value we find must always be compared with the conditions already established. Here $7 > 6$: it is acceptable.

1.4 — The equation of the ellipse. With $a = 7$:

$$4x^2 + y^2 = 4 \implies \boxed{\mathcal{E} : \ x^2 + \frac{y^2}{4} = 1}$$

Dividing by 4 we recover the canonical form. It also agrees with the general formula: $s^2 = \frac{7-3}{7-6} = 4$. Let us verify that the given point lies on $\mathcal{E}$: $\left(\frac{1}{2}\right)^2 + \frac{(\sqrt{3})^2}{4} = \frac{1}{4} + \frac{3}{4} = 1$. ✓

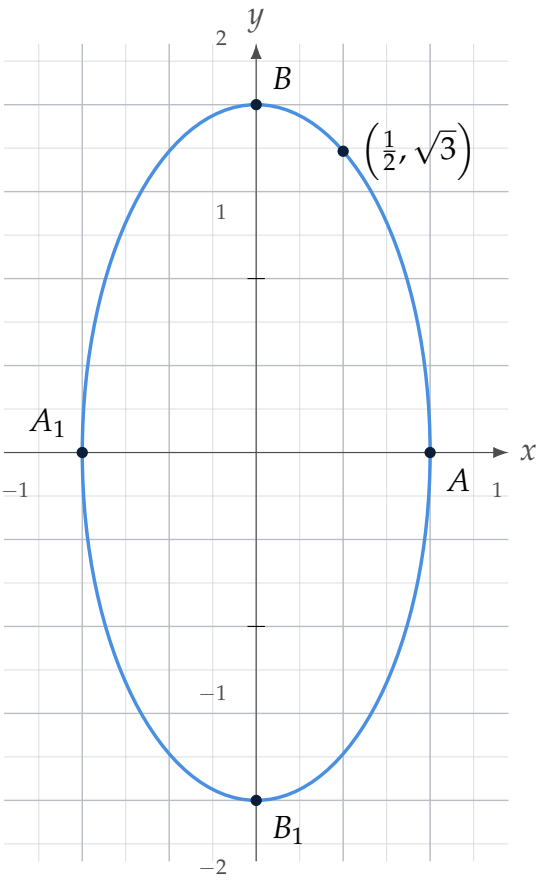


Figure 1. The ellipse $x^2 + \frac{y^2}{4} = 1$ with vertices A, B, A_1, B_1 and the point $\left(\frac{1}{2}, \sqrt{3}\right)$.

1.5 — The vertices. Here $p^2 = 1$ and $s^2 = 4$, so the semi-axes are $p = 1$ and $s = 2$. By M1 the vertices are $(\pm 1, 0)$ and $(0, \pm 2)$. Naming them counterclockwise starting from A, which has positive x-coordinate (Figure 1):

$$A(1,0), \qquad B(0,2), \qquad A_1(-1,0), \qquad B_1(0,-2).$$

Let's recap — Step 1

- Passing through $\left(\frac{1}{2}, \sqrt{3}\right)$ fixes $a = 7$.
- The ellipse is $\mathcal{E}: \; x^2 + \dfrac{y^2}{4} = 1$, with semi-axes $p = 1$ and $s = 2$.
- The vertices are $A(1,0)$, $B(0,2)$, $A_1(-1,0)$, $B_1(0,-2)$.

Part a — The inscribed square

Where we are. We have the ellipse $\mathcal{E}: \; x^2 + \frac{y^2}{4} = 1$ and its vertices.

Goal. Find the area of the square inscribed in $\mathcal{E}$ with sides parallel to the axes.

Idea. A square is inscribed in the ellipse when its four vertices lie on the ellipse. We exploit **symmetry** (Memento M3, below): it is enough to know one vertex, the others are obtained by changing the signs of the coordinates. Then we impose two conditions: that it is a square, and that the vertex lies on $\mathcal{E}$.

> **Memento M3 — Symmetries of a curve with respect to the axes**
> If in an equation, replacing x with $-x$, the equation does not change, the curve is **symmetric with respect to the y-axis**: together with every point (x, y) it also contains $(-x, y)$.
> If replacing y with $-y$ the equation does not change, the curve is **symmetric with respect to the x-axis**: together with every point (x, y) it also contains $(x, -y)$.
> If both hold, the curve also contains $(-x, -y)$.

a.1 — The four vertices of the square. In the equation of $\mathcal{E}$ the letters x and y appear only squared, so both symmetries of M3 hold. If $(x, y) \in \mathcal{E}$, then $(-x, y)$, $(x, -y)$ and $(-x, -y)$ also lie on $\mathcal{E}$.

Let $C(x_C, y_C)$ be the vertex of the square in the first quadrant, with $x_C > 0$ and $y_C > 0$. The other three, counterclockwise, are

$$D(-x_C,\, y_C), \qquad E(-x_C,\, -y_C), \qquad F(x_C,\, -y_C).$$

The four points form a rectangle with sides parallel to the axes, of dimensions $2x_C$ and $2y_C$.

Why must the vertices have this form? Two vertices on the same horizontal side have the same y-coordinate y_0. The points of $\mathcal{E}$ with that y-coordinate satisfy $x^2 = 1 - \frac{y_0^2}{4}$, hence opposite x-coordinates, $\pm x_0$. The same argument on the vertical sides shows that the other two vertices have y-coordinates $\pm y_0$. It follows that the vertices always have the form $(\pm x_C, \pm y_C)$ (Figure 2).

a.2 — The two conditions. The rectangle is a square if its sides are equal, that is, $2x_C = 2y_C$, so $x_C = y_C$. Moreover, C must lie on the ellipse. We put the two conditions into a system:

$$\begin{cases} x_C = y_C \\ x_C^2 + \dfrac{y_C^2}{4} = 1 \end{cases}$$

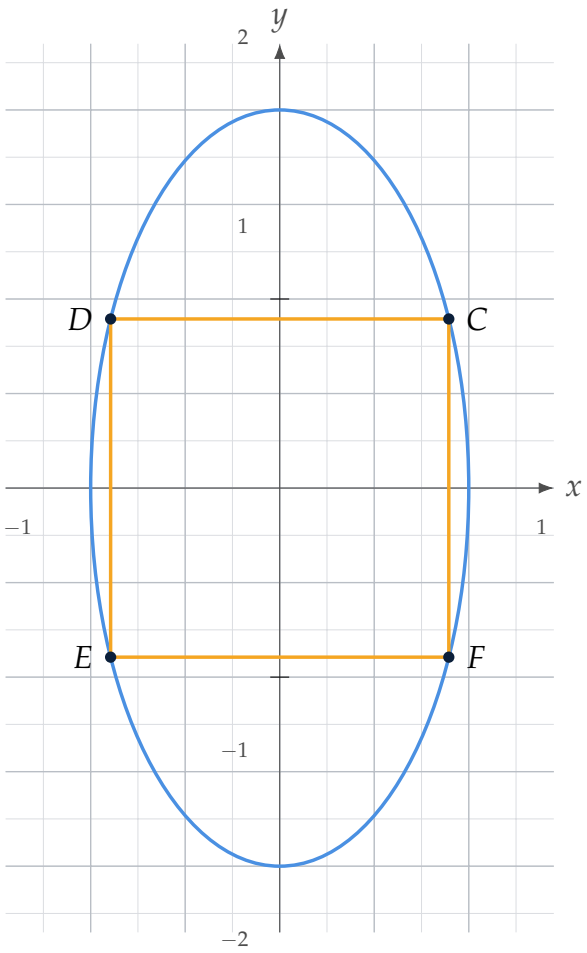


Figure 2. The ellipse with the inscribed rectangle $CDEF$ and the coordinates of its vertices.

a.3 — We solve. We substitute $y_C = x_C$ into the second equation:

$$\begin{aligned} x_C^2 + \frac{x_C^2}{4} = 1 &\implies \frac{4x_C^2 + x_C^2}{4} = 1 \\ &\implies 5x_C^2 = 4 \\ &\implies x_C^2 = \frac{4}{5}. \end{aligned}$$

Since C is in the first quadrant, we take the positive root (the negative one would correspond to vertex E):

$$x_C = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$

a.4 — The area. The side is $L = 2x_C = \dfrac{4\sqrt{5}}{5}$ and the area is

$$L^2 = \frac{16 \cdot 5}{25} = \boxed{\frac{16}{5}}.$$

We reach the same result more quickly: $L^2 = (2x_C)^2 = 4x_C^2 = 4 \cdot \frac{4}{5} = \frac{16}{5}$.

Check. The vertex $C\left(\frac{2\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$ satisfies the equation, because $\frac{4}{5} + \frac{4/5}{4} = \frac{4}{5} + \frac{1}{5} = 1$. ✓ As an order of magnitude, $\frac{16}{5} = 3.2$ is smaller than the area of the ellipse ($\pi \cdot 1 \cdot 2 \approx 6.28$), as it must be for a square contained in it.

Let's recap — Part a

- By symmetry the vertices of the square are $(\pm x_C, \pm y_C)$: it is enough to find one.
- Square $\Rightarrow x_C = y_C$; the vertex lies on $\mathcal{E} \Rightarrow x_C^2 = \frac{4}{5}$.
- The side is $2x_C = \frac{4\sqrt{5}}{5}$ and the area is $\frac{16}{5}$.

Part b — The parabola with vertex at A

Where we are. We know the vertices of the ellipse: $A(1,0)$, $B(0,2)$, $B_1(0,-2)$.

Goal. Find the equation of the parabola with vertex at A and passing through B and B_1.

Idea. First we choose the **type** of parabola (vertical or horizontal axis) by looking at the given points, then we impose the conditions on the coefficients.

Memento M4 — The parabola with axis parallel to the x-axis

Parabolas of equation $y = \alpha x^2 + \beta x + \gamma$ (with $\alpha \neq 0$) have their axis of symmetry parallel to the y-axis and are graphs of functions. Parabolas of equation

$$x = \alpha y^2 + \beta y + \gamma, \qquad \alpha \neq 0$$

have instead their axis parallel to the x-axis. For the latter, the vertex is

$$V\left(-\frac{\Delta}{4\alpha}, -\frac{\beta}{2\alpha}\right), \qquad \Delta = \beta^2 - 4\alpha\gamma \text{ (the discriminant).}$$

The vertex belongs to the parabola.

(In textbooks the coefficients are a, b, c. Here we use α, β, γ because a is already the parameter of the problem.)

b.1 — Which type of parabola? The points $B(0,2)$ and $B_1(0,-2)$ have the **same x-coordinate** but different y-coordinates. A parabola of the type $y = \alpha x^2 + \dots$ is the graph of a function, and a function assigns to each x only one value of y: it cannot contain two points on the same vertical line. We therefore need the form with a horizontal axis (M4):

$$x = \alpha y^2 + \beta y + \gamma.$$

Watch out — Don't write $y = \alpha x^2 + \beta x + \gamma$ out of habit

Before writing the equation of a parabola, check whether the data (points, vertex, axis) are compatible with a vertical axis. Two points with the same x-coordinate, like B and B_1, rule it out.

b.2 — We impose the conditions.

1. **The vertex is $A(1,0)$.** Its y-coordinate, $-\frac{\beta}{2\alpha}$, must equal 0, so $\beta = 0$ (the equation becomes $x = \alpha y^2 + \gamma$). Moreover, the vertex belongs to the parabola: substituting $x = 1, y = 0$ we get $1 = \gamma$.

2. **Passing through $B(0,2)$.** We substitute $x = 0, y = 2$ into $x = \alpha y^2 + 1$:

$$0 = 4\alpha + 1 \implies \alpha = -\frac{1}{4}.$$

3. **Passing through $B_1(0,-2)$.** No new condition is needed: the axis of the parabola is the line $y = 0$, and B_1 is the mirror image of B across this axis. A direct check confirms it: $-\frac{1}{4} \cdot (-2)^2 + 1 = -1 + 1 = 0$. ✓

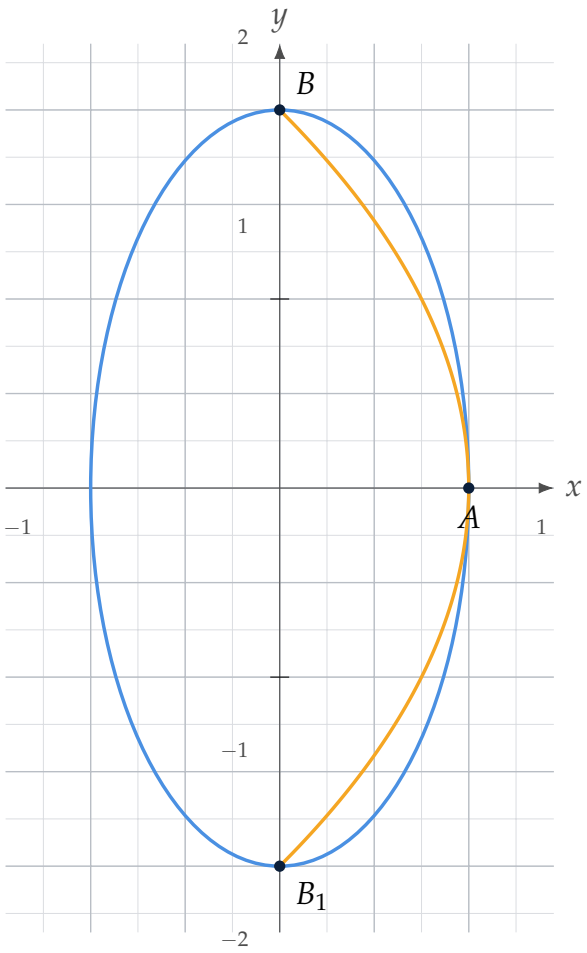


Figure 3. The parabola $x = -\frac{1}{4}y^2 + 1$ with vertex at A, passing through B and B_1, drawn over the ellipse.

$$\boxed{\mathcal{P}:\ x = -\frac{1}{4}y^2 + 1}$$

b.3 — Checking the vertex. With $\alpha = -\frac{1}{4}, \beta = 0, \gamma = 1$ we have $\Delta = 0 - 4 \cdot \left(-\frac{1}{4}\right) \cdot 1 = 1$. The vertex is

$$V\left(-\frac{1}{4 \cdot (-\frac{1}{4})}, 0\right) = (1, 0) = A. \checkmark$$

Since $\alpha < 0$, the parabola opens toward negative x: A is its rightmost point and the curve passes through B and B_1 (Figure 3).

Let's recap — Part b

- B and B_1 have the same x-coordinate $\Rightarrow$ the parabola has a horizontal axis: $x = \alpha y^2 + \beta y + \gamma$.
- Vertex at $A(1,0) \Rightarrow \beta = 0$, $\gamma = 1$. Passing through $B \Rightarrow \alpha = -\frac{1}{4}$.
- $\mathcal{P}: \ x = -\frac{1}{4}y^2 + 1$, with axis $y = 0$, opening toward negative x.

Historical note — Where the names "ellipse", "parabola", "hyperbola" come from

The conics, curves obtained by cutting a cone with a plane, were known to Greek mathematicians even before Apollonius of Perga (3rd century BC). Apollonius, author of the work *Conics*, is traditionally credited with introducing the names we still use today. They are Greek words that express a comparison: *élleipsis* means "deficiency", *parabolé* "equality" or "application", *hyperbolé* "excess". The comparison was between an area associated with the curve and a reference area: depending on the conic, the first turned out smaller than, equal to, or greater than the second — hence the three names. In our problem we meet exactly two of these three curves: the ellipse (Steps 0 and 1) and the parabola (part b).

Part c — The family of lines and the arc of the parabola

Where we are. We have the parabola $\mathcal{P}: \ x = -\frac{1}{4}y^2 + 1$, with $A(1,0)$ and $B(0,2)$.

Goal. Find for which $k \neq 0$ the lines of equation $4kx + ky = k - 4$ have at least one point in common with the part of $\mathcal{P}$ lying in the first quadrant (the *arc*).

Idea. Let us slide the lines of the family along the arc and look for the limiting positions: the line through B, the line through A and the tangent line.

c.1 — Recognizing the family of lines

Memento M5 — Equation of a line and slope

A line can be written in **implicit** form, $ux + vy + w = 0$, or in **explicit** form, $y = mx + q$. To go from one to the other, if $v \neq 0$:

$$vy = -ux - w \implies y = -\frac{u}{v}x - \frac{w}{v},$$

so the **slope** is $m = -\frac{u}{v}$ and the y**-intercept** (the y-coordinate of the point where the line crosses the y-axis) is $q = -\frac{w}{v}$. Two lines are parallel if and only if they have the same m.

(Here too we use letters other than a, b, c.)

Family of lines. A family of lines is a set of infinitely many lines whose equation depends on a parameter (here k). (In more advanced texts you may also meet the word *pencil*.)

- **Family of concurrent lines** (in Italian, *fascio proprio*): all the lines pass through the same point, the *center*.
- **Family of parallel lines** (in Italian, *fascio improprio*): all the lines are parallel.

We move everything to the left: $4kx + ky - k + 4 = 0$, that is, $u = 4k$, $v = k$, $w = 4 - k$.

Why $k \neq 0$? If $k = 0$ the equation would become $0 = -4$, which is false for every (x, y): it describes no point of the plane. Moreover, $k \neq 0$ allows us to divide by k.

By M5 the slope is

$$m = -\frac{4k}{k} = -4 \quad \text{for every } k \neq 0,$$

so **all the lines of the family have the same** m: they are parallel, so this is a **family of parallel lines**. As k varies in $\mathbb{R} \setminus \{0\}$, the slope stays constant while the y-intercept is given by

$$q = -\frac{w}{v} = \frac{k-4}{k} = 1 - \frac{4}{k}.$$

Every line of the family can therefore be written $y = -4x + q$ with $q = 1 - \frac{4}{k}$. The value $q = 1$ is never obtained, because it would require $\frac{4}{k} = 0$; for our problem this is irrelevant, as we will see. Every other value of q corresponds to exactly one k.

c.2 — The arc of the parabola

The parabola is $x = 1 - \frac{y^2}{4}$. The first quadrant is the region with $x \geq 0$ and $y \geq 0$; we also include the points on the axes, hence also A and B.

- The condition $y \geq 0$ is already a requirement.
- The condition $x \geq 0$ gives $1 - \frac{y^2}{4} \geq 0$, that is, $y^2 \leq 4$, that is, $y \leq 2$ (with $y \geq 0$).

The arc is therefore the portion of $\mathcal{P}$ with $0 \leq y \leq 2$, going from $A(1,0)$ to $B(0,2)$.

c.3 — The three special lines

All the lines of the family have slope -4: we imagine **sliding** them parallel to themselves, from lower left to upper right. We ask in which positions they touch the arc. To answer, three special lines are enough, and for each we compute the value of k.

The line through $B(0,2)$. We substitute the coordinates of B into the equation of the family:

$$4k \cdot 0 + 2k = k - 4 \implies 2k = k - 4 \implies k = -4.$$

(Consistent with $q = 2$: $1 - \frac{4}{-4} = 2$.)

The line through $A(1,0)$.

$$4k \cdot 1 + 0 = k - 4 \implies 3k = -4 \implies k = -\frac{4}{3}.$$

(Consistent with $q = 4$.)

The line tangent to the arc. We put the line and the parabola into a system and study when there is exactly one solution.

> **Memento M6 — Line and parabola: relative position**
> If we substitute the equation of the parabola into that of the line, we obtain a quadratic equation. The sign of its discriminant Δ tells us how the two curves are placed:
>
> - $\Delta > 0$: two intersection points (the line is a **secant**);
> - $\Delta = 0$: a single point, a double one (the line is **tangent**);
> - $\Delta < 0$: no points (the line is **external**).

We substitute $x = 1 - \frac{y^2}{4}$ into the equation of the family:

$$4k\left(1 - \frac{y^2}{4}\right) + ky = k - 4 \implies 4k - ky^2 + ky = k - 4$$
$$\implies -ky^2 + ky + 3k + 4 = 0.$$

We multiply by -1 (the solutions do not change) to remove the leading minus sign:

$$ky^2 - ky - 3k - 4 = 0.$$

By M6 we compute the discriminant:

$$\Delta = (-k)^2 - 4 \cdot k \cdot (-3k - 4) = k^2 + 12k^2 + 16k = 13k^2 + 16k = k(13k + 16).$$

We impose $\Delta = 0$:

$$k(13k + 16) = 0 \implies k = 0 \text{ or } k = -\frac{16}{13}.$$

The value $k = 0$ is not allowed, so the tangent corresponds to $k = -\frac{16}{13}$.

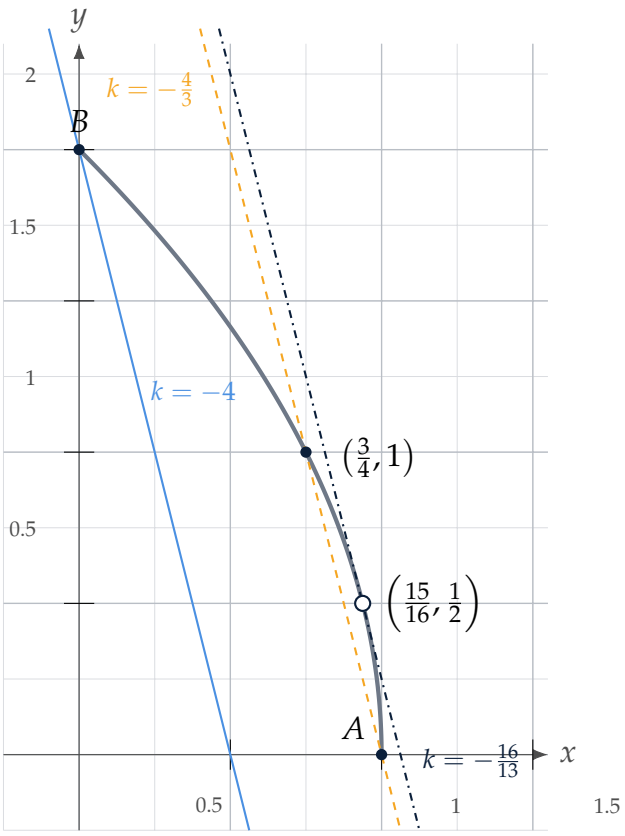
Where does it touch the parabola? With $\Delta = 0$ the double solution is $y = \frac{k}{2k} = \frac{1}{2}$, so $x = 1 - \frac{1}{16} = \frac{15}{16}$. The point of tangency is $\left(\frac{15}{16}, \frac{1}{2}\right)$ and it has $0 \le y \le 2$: it lies on the arc (Figure 4). In terms of the intercept, $q = 1 - \frac{4}{-16/13} = 1 + \frac{13}{4} = \frac{17}{4}$.

> **Watch out — The point of tangency must lie on the arc**
> The tangent to the *whole* parabola might touch it at a point outside the arc: in that case it would not be a limiting position for our problem. After finding k, always check where the point of tangency falls.

c.4 — From the family to the interval of k

For $k < 0$, as k increases toward 0 the ratio $\frac{4}{k}$ becomes more and more negative, so $q = 1 - \frac{4}{k}$ **increases**: sliding the line upward along the arc corresponds to increasing k (while it stays negative).

- The leftmost line that touches the arc is the one through B ($k = -4$): the lines with $k < -4$ pass outside the arc.
- As k increases, the line meets the arc, until it reaches the tangent position ($k = -\frac{16}{13}$), which is the last one.
- Beyond the tangent ($-\frac{16}{13} < k < 0$) no line touches the arc any more.



through B ($k = -4$) through A ($k = -\frac{4}{3}$) tangent ($k = -\frac{16}{13}$)

Figure 4. The arc AB with the three lines of slope -4: through B ($k = -4$), through A ($k = -\frac{4}{3}$) and tangent ($k = -\frac{16}{13}$, point of tangency $\left(\frac{15}{16}, \frac{1}{2}\right)$).

- For $k > 0$ the line is even further "to the left" than the one through B: it does not touch the arc.

The lines that meet the arc are those with $-4 \leq k \leq -\frac{16}{13}$. The endpoints are included: for $k = -4$ the line passes through B, which is a point of the arc, and for $k = -\frac{16}{13}$ the line is tangent to the arc (one point in common).

$$\boxed{-4 \leq k \leq -\frac{16}{13}}$$

Watch out — The endpoint is not the line through A

A common mistake is to think that, since the arc is bounded by A and B, the endpoints of the interval are the lines through A and through B, that is, $-4 \leq k \leq -\frac{4}{3}$. That would be true for a segment, not for a **curved** arc. The tangent line touches the arc "further along" than the one through A: $-\frac{4}{3} \approx -1.333$ lies *inside* the correct interval, because $-\frac{16}{13} \approx -1.231$ is larger. The lines with k between $-\frac{4}{3}$ and $-\frac{16}{13}$ do meet the arc, even though they meet it at points other than A and B.

A practical rule: when the object is curved, besides the lines through the endpoints you must always look for the **tangent**.

c.5 — Checking with test values

Let us verify the answer with a few values of k: one for each excluded zone (-5, -1, 1), the three notable values, and one interior value (-2). The equation to solve is $ky^2 - ky - 3k - 4 = 0$; a solution is valid for the arc only if $0 \leq y \leq 2$. In the table the equations are already simplified.

k	equation in y	solutions y	solutions on the arc	conclusion
-5	$5y^2 - 5y - 11 = 0$	≈ 2.07 and -1.07	none	does not meet the arc
-4	$y^2 - y - 2 = 0$	2 and -1	$y = 2$, i.e. B	meets it (only at B)
-2	$y^2 - y - 1 = 0$	≈ 1.62 and -0.62	$y \approx 1.62$	meets it (one point)
$-\frac{4}{3}$	$y^2 - y = 0$	0 and 1	$y = 0$ (A) and $y = 1$	meets it (two points)
$-\frac{16}{13}$	$\Delta = 0$	$\frac{1}{2}$ (double)	$y = \frac{1}{2}$	tangent
-1	$y^2 - y + 1 = 0$	no real solutions	none	does not meet the arc
1	$y^2 - y - 7 = 0$	≈ 3.19 and -2.19	none	does not meet the arc

Two useful remarks:

- for $k = -5$ and $k = 1$ the line meets the **parabola**, **but at points outside the arc**: $\Delta > 0$ alone is not enough, we must check that the solutions lie on the arc;
- for $k = -\frac{4}{3}$ the line through A meets the arc at two points, A and $\left(\frac{3}{4}, 1\right)$: it is an interior value, not an endpoint.

Let's recap — Part c

- The family is **made of parallel lines**: all have $m = -4$ and $q = 1 - \frac{4}{k}$.
- The arc is the part of $\mathcal{P}$ with $0 \le y \le 2$, from A to B.
- Limiting positions: through B ($k = -4$) and tangent ($k = -\frac{16}{13}$, point $\left(\frac{15}{16}, \frac{1}{2}\right)$ on the arc). The line through A ($k = -\frac{4}{3}$) lies inside the interval.
- Result: $-4 \le k \le -\frac{16}{13}$.

Summary

step	result
Step 0	ellipse with foci on $y \iff a > 6$
Step 1	$a = 7$; $\mathcal{E}: \; x^2 + \frac{y^2}{4} = 1$; $A(1,0)$, $B(0,2)$, $A_1(-1,0)$, $B_1(0,-2)$
a)	area of the inscribed square: $\frac{16}{5}$
b)	$\mathcal{P}: \; x = -\frac{1}{4}y^2 + 1$
c)	$-4 \le k \le -\frac{16}{13}$

About the author. Nicola Maglione is an electronic engineer (degree in Electronic Engineering, University of Naples, 2001), with over twenty years of experience in the semiconductor industry as a design and program manager, specialized in NAND Flash memory and high-speed interfaces, and holder of several patents in memory architectures. Alongside his professional work, he has always cultivated a passion for mathematics — particularly Analysis, Abstract Algebra, and Number Theory — and for physics, which he shares through keenonscience, his science-outreach page, and on the website keenonmaths.org.

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