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Equivalent Resistance of an Infinite Ladder Network

Nihil sub sole novum

Self-consistency, continued fractions, and the Golden Ratio

An in-depth guide bridging circuit theory and number theory

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Introduction

The problem of determining the equivalent resistance R_{eq} of an **infinite ladder network** is a classic example of circuit analysis, with applications in transmission lines and the modelling of composite materials. Consider a network built from an infinite repetition of "L" cells, each consisting of:

- a **series** resistor of value R;
- a **shunt** (parallel) resistor, connected to the return rail, of value $\mathbf{R_p = pR}$, where p is a positive real coefficient ($p > 0$).

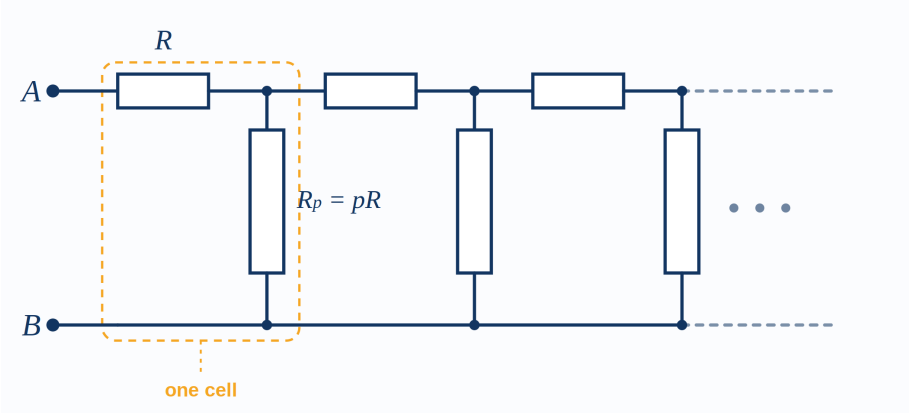

Figure 1: *Diagram of the infinite ladder network: each cell consists of a series resistor R and a shunt resistor $R_p = pR$.*

The goal is to compute the resistance seen at the input terminals A-B. We will follow two paths, independent of each other but converging to the same result:

- **Method 1 (Sections 2–3) – Self-consistency**: exploiting the fact that the network is infinite, we show that R_{eq} must satisfy a fixed-point equation, whose only physically admissible solution is $R_{eq} = \dfrac{R}{2}\left(1 + \sqrt{1 + 4p}\right)$.
- **Method 2 (Section 5) – Continued fractions and Fibonacci**: in the particular case $p = 1$, we show that the resistance of the network truncated at n stages is the ratio of two consecutive Fibonacci numbers, and that the limit as $n \to \infty$ coincides with the Golden Ratio.

The case $p = 1$ (Section 4) is the historically best-known one: the constant that emerges is the **Golden Ratio** $\varphi \approx 1.618$, an unexpected bridge between circuit theory and number theory. Section 6 explores what happens for arbitrary p, and a final appendix looks more closely at the connection with the family of "metallic numbers". *Scope.* We deal exclusively with the real resistive case (DC regime). The extension to complex impedances (AC regime) is material for a separate article.

1 Review: resistive networks and the network truncated at n stages

Review (series/parallel combination). *Two resistors R_1, R_2 in series give total resistance $R_1 + R_2$; in parallel they give $R_1 \parallel R_2 = \dfrac{R_1 R_2}{R_1 + R_2}$.*

Definition 1.1 (Network truncated at n stages). *We call the **network truncated at n stages** the finite network obtained by keeping only the first n "L" cells starting from terminals A-B, and terminating beyond the last cell with an open circuit (that is: the last shunt resistor R_p is the final element of the network, nothing follows it). We denote by R_n the resistance seen at terminals A-B of this finite network.*

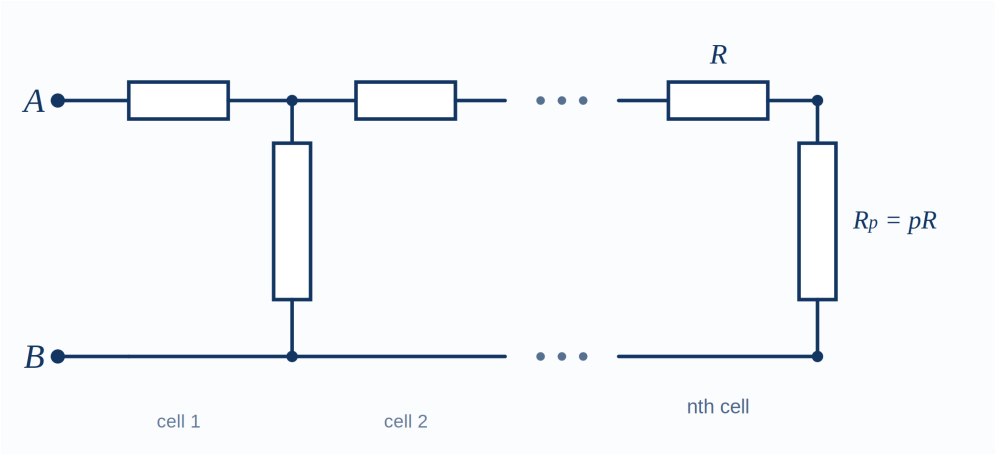

Figure 2: *Network truncated at n stages: it stops after the n-th cell with an open circuit. R_n is the resistance seen at terminals A-B.*

For $n = 1$ (a single cell, no network beyond it), the parallel combination is with an open circuit, so the resistor R_p is simply the last series element:

$$R_1 = R + R_p = R(1 + p).$$

For $n \geq 2$, the first cell sees, in place of the plain R_p, the combination of R_p in parallel with the entire remaining subnetwork (which is itself a network truncated at $n - 1$ stages, of resistance R_{n-1}):

Recursive relation:

$$R_n = R + (R_p \parallel R_{n-1}) = R + \frac{R_p \cdot R_{n-1}}{R_p + R_{n-1}}, \qquad n \geq 2, \qquad R_1 = R(1 + p).$$

This relation is the starting point both for Method 1 (Sections 2–3, where we study the behaviour of (R_n) as $n \to \infty$) and for Method 2 (Section 5, where for $p = 1$ we recognize in (R_n) the sequence of Fibonacci ratios). It is convenient to work in dimensionless form: set $r_k := R_k/R$ for every $k \in \{1, \ldots, n\}$ (in particular, r_k is a pure, dimensionless number). We divide both sides of the recursive relation by R, and substitute $R_p = pR$ and $R_{n-1} = r_{n-1}R$:

$$\frac{R_n}{R} = 1 + \frac{1}{R} \cdot \frac{R_p \cdot R_{n-1}}{R_p + R_{n-1}} = 1 + \frac{1}{R} \cdot \frac{(pR)(r_{n-1}R)}{pR + r_{n-1}R} = 1 + \frac{p\, r_{n-1}}{p + r_{n-1}},$$

Likewise, from the base case $R_1 = R(1+p)$ we get $r_1 = 1+p$. The recursive relation thus becomes

$$r_n = 1 + \frac{p\, r_{n-1}}{p + r_{n-1}}, \qquad n \geq 2, \qquad r_1 = 1+p. \tag{1}$$

2 Existence and uniqueness of R_{eq}

This section provides a rigorous foundation for what Method 1 (Section 3) does in a purely formal way.

Definition 2.1. *Let (with $p > 0$ fixed)*

$$f : \; x \in (0,\infty) \;\longmapsto\; f(x) = 1 + \frac{px}{p+x} \;\in (0,\infty).$$

Relation (1) can then be written as $r_n = f(r_{n-1})$.

Remark. f is well defined and positive-valued: for $x > 0$ we have $p + x > 0$ and $f(x) > 1 > 0$.

Lemma 2.1 (monotonicity of f). *f is strictly increasing on $(0,\infty)$.*

Proof. $f'(x) = \dfrac{p(p+x) - px}{(p+x)^2} = \dfrac{p^2}{(p+x)^2} > 0$ for every $x > 0$. ■ *Remark (a common pitfall).* Knowing that f is increasing does **not** yet tell us whether the sequence (r_n), defined by $r_n = f(r_{n-1})$, is itself increasing or decreasing: the monotonicity of f and the monotonicity of the sequence it generates are two distinct things. What governs the direction of the iteration is the sign of the auxiliary function $g(x) := f(x) - x$: if $g(x) < 0$ one step of the iteration makes the value *decrease* ($f(x) < x$); if $g(x) > 0$ it makes it *increase*. Let us study g.

Lemma 2.2 (fixed point). *Let $g(x) := f(x) - x$. Then g is strictly decreasing on $(0,\infty)$, and the equation $f(x) = x$ has, among positive solutions, a unique solution*

$$r^* = \frac{1 + \sqrt{1+4p}}{2}.$$

Moreover $g(x) < 0$ for $x > r^*$ and $g(x) > 0$ for $0 < x < r^*$.

Proof. First, $g'(x) = f'(x) - 1 = \dfrac{p^2}{(p+x)^2} - 1 < 0$ for every $x > 0$ (since $p + x > p$, so $(p+x)^2 > p^2$): g is therefore strictly decreasing on $(0,\infty)$, and in particular **injective**, so it has **at most one** zero. Let us now look for the zeros of g, i.e. the solutions of $f(x) = x$: $1 + \dfrac{px}{p+x} = x \iff (p+x) + px = x(p+x) \iff x^2 - x - p = 0$ (algebraic details in Section 3). The two roots are $\dfrac{1 \pm \sqrt{1+4p}}{2}$. Since $p > 0$, we have $\sqrt{1+4p} > 1$, so the root $\dfrac{1 - \sqrt{1+4p}}{2}$ is **negative** and must be discarded: the only zero of g in the domain $(0,\infty)$ is $r^* = \dfrac{1 + \sqrt{1+4p}}{2}$. Since g is strictly decreasing with $g(r^*) = 0$, it follows immediately that $g(x) < 0$ for $x > r^*$ and $g(x) > 0$ for $x < r^*$. ■

Lemma 2.3. *$r_1 > r^*$, and $r_n > r^*$ for every $n \geq 1$.*

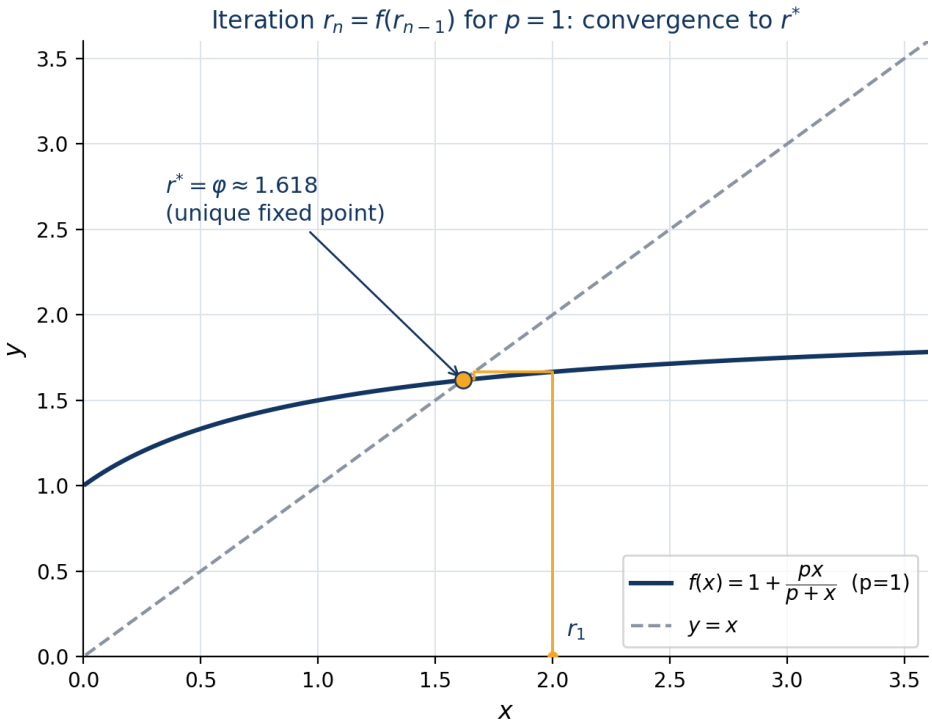

Figure 3: *Graph of f for $p = 1$ and of the line $y = x$: the only intersection on the positive half-line is $r^* = \varphi$ (Lemma 2.2). The staircase path shows the iteration $r_n = f(r_{n-1})$ starting from r_1, which converges to r^* (Theorem 2.4).*

Proof. $r_1 = 1 + p$; let us compare it with r^*: since $(1+2p)^2 = 1 + 4p + 4p^2 > 1 + 4p$ for $p > 0$, we have $1 + 2p > \sqrt{1+4p}$, hence $r^* = \dfrac{1+\sqrt{1+4p}}{2} < \dfrac{1+(1+2p)}{2} = 1 + p = r_1$. So $r_1 > r^*$. By induction: if $r_{n-1} > r^*$, then by the monotonicity of f (Lemma 2.1 – *this* is where it is really needed) and since $f(r^*) = r^*$ (Lemma 2.2), we get $r_n = f(r_{n-1}) > f(r^*) = r^*$. ■

Theorem 2.4 (existence and uniqueness of R_{eq}). *The sequence (r_n) is strictly decreasing and converges to $r^* = \dfrac{1+\sqrt{1+4p}}{2}$. Consequently $R_{eq} = \lim_{n\to\infty} R_n = R\,r^* = \dfrac{R}{2}\left(1+\sqrt{1+4p}\right)$ exists, is unique, and is the only physically admissible value (the other root of the quadratic is negative, incompatible with a resistance).*

Proof. By Lemma 2.3, $r_n > r^*$ for every n; by Lemma 2.2, $g(x) < 0$ for every $x > r^*$; combining the two facts, $g(r_n) < 0$ for every n, i.e.

$$r_{n+1} = f(r_n) < r_n \qquad \text{for every } n :$$

this is exactly the direction of the iteration hinted at in the remark after Lemma 2.1 – here we obtain its rigorous confirmation. The sequence (r_n) is therefore strictly decreasing, and by Lemma 2.3 it is also bounded below by r^*. By the **monotone convergence theorem**, it converges to some limit $L \geq r^*$. Taking the limit in the relation $r_{n+1} = f(r_n)$, and by continuity of f, we obtain $L = f(L)$: L is a positive fixed point of f, and by the uniqueness established in Lemma 2.2, $L = r^*$. ■

3 The self-consistency method

Since the network is infinite, removing the first cell does not change the total resistance (a self-similarity property): the network seen after the first cell is indistinguishable from the original network. Hence R_{eq} must satisfy

$$R_{eq} = R + (R_p \parallel R_{eq}) = R + \frac{pR \cdot R_{eq}}{pR + R_{eq}}. \tag{2}$$

(Section 2 has just shown that this equation genuinely makes sense: it is the fixed-point equation to which (R_n) converges, and its positive solution exists and is unique.) **Solving.** Multiplying both sides of (2) by $(pR + R_{eq})$:

$$\begin{aligned} R_{eq}(pR + R_{eq}) &= R(pR + R_{eq}) + pR \cdot R_{eq} \\ pR \cdot R_{eq} + R_{eq}^2 &= pR^2 + R \cdot R_{eq} + pR \cdot R_{eq} \\ R_{eq}^2 &= pR^2 + R \cdot R_{eq} \\ R_{eq}^2 - R \cdot R_{eq} - pR^2 &= 0. \end{aligned} \tag{3}$$

A quadratic equation in R_{eq}, with $a = 1$, $b = -R$, $c = -pR^2$:

$$R_{eq} = \frac{R \pm \sqrt{R^2 + 4pR^2}}{2} = \frac{R}{2}\left(1 \pm \sqrt{1 + 4p}\right).$$

The root with the $-$ sign is negative (Section 2, Lemma 2.2) and must be discarded on physical grounds. What remains is the result we were after:

$$\boldsymbol{R_{eq} = \frac{R}{2}\left(1 + \sqrt{1 + 4p}\right)} \tag{4}$$

4 The case $p = 1$: the Golden Ratio

For $p = 1$ (i.e. $R_p = R$: the shunt resistance equal to the series one), (4) becomes

$$\boldsymbol{R_{eq} = \frac{R}{2}(1 + \sqrt{5}) = R\varphi}$$

where $\varphi = \dfrac{1 + \sqrt{5}}{2} \approx 1.6180339887\ldots$ is the **Golden Ratio**. *Remark.* Notable properties of φ, all direct consequences of the equation $\varphi^2 - \varphi - 1 = 0$ (the $p = 1$ case of (3), in dimensionless form):

- $\varphi^2 = \varphi + 1$;
- $\dfrac{1}{\varphi} = \varphi - 1 \approx 0.618\ldots$ (obtained by dividing $\varphi^2 = \varphi + 1$ by φ: $\varphi = 1 + \dfrac{1}{\varphi}$, hence $\dfrac{1}{\varphi} = \varphi - 1$);
- $\varphi = 1 + \dfrac{1}{\varphi}$ – this identity is exactly the reason why φ admits the periodic continued-fraction representation $[1; 1, 1, 1, \ldots]$, the subject of Section 5.

5 Continued fractions and the Fibonacci sequence

Definition 5.1 (simple continued fraction). *Given a sequence of positive integers $a_0, a_1, a_2, \ldots$, the finite continued fraction of order n is*

$$[a_0; a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

*The values $[a_0; a_1, \ldots, a_n]$ are called **convergents**. An infinite continued fraction $[a_0; a_1, a_2, \ldots]$ is defined, when it exists, as the limit of its convergents as $n \to \infty$.*

The case relevant here: $[1; 1, 1, 1, \ldots]$**.** Set $x_n = [1; \underbrace{1, \ldots, 1}_{n-1}]$ (n terms in total). One checks that $x_1 = 1$ and, for $n \geq 2$,

$$x_n = 1 + \frac{1}{x_{n-1}} =: h(x_{n-1}), \qquad h(x) := 1 + \frac{1}{x}.$$

Unlike f in Section 2, h is **decreasing** on $(0, \infty)$ ($h'(x) = -1/x^2 < 0$): the monotonicity argument used there does not apply directly. A different argument is needed, classical in the study of continued fractions.

Proposition 5.1 (convergence). *(x_n) converges, and its limit is φ.*

Proof. *Step 1 – the sequence stays in $[1, 2]$.* By induction: $x_1 = 1 \in [1, 2]$; if $x_{n-1} \in [1, 2]$ then

$$x_n = 1 + \frac{1}{x_{n-1}} \;\in\; [1.5,\, 2] \;\subset\; [1, 2].$$

Step 2 – why it helps to look at $h \circ h$. h is decreasing, so every application of h reverses order: $a < b$ implies $h(a) > h(b)$. Applying it twice restores the order, i.e. $h \circ h$ is **increasing**:

$$(h \circ h)(x) = h\left(1 + \frac{1}{x}\right) = 1 + \frac{1}{1 + 1/x} = \frac{2x + 1}{x + 1}, \qquad \frac{d}{dx}\left(\frac{2x+1}{x+1}\right) = \frac{1}{(x+1)^2} > 0.$$

For this reason we separate the odd-indexed terms from the even-indexed ones,

$$y_k := x_{2k-1}, \qquad z_k := x_{2k}, \qquad k = 1, 2, 3, \ldots,$$

which then advance two steps at a time via the increasing function $h \circ h$:

$$y_{k+1} = (h \circ h)(y_k), \qquad z_{k+1} = (h \circ h)(z_k).$$

Step 3 – the only fixed point of $h \circ h$ in $[1, 2]$.

$$(h \circ h)(x) = x \iff \frac{2x+1}{x+1} = x \iff x^2 - x - 1 = 0 \iff x = \varphi \;\text{ or }\; x = \frac{1 - \sqrt{5}}{2}.$$

The second root is negative, so φ is the only fixed point of $h \circ h$ in $[1, 2]$ (in fact, in all of $(0, \infty)$). *Step 4 – which side of φ do y_k and z_k lie on.* Here we need the same type of argument as in

Section 2: the mere fact that $h \circ h$ is increasing is not enough to tell us whether (y_k) or (z_k) are monotone; we need the sign of $k(x) := (h \circ h)(x) - x$. From Step 3,

$$k(x) \;=\; \frac{2x+1}{x+1} - x \;=\; \frac{-(x^2 - x - 1)}{x+1},$$

and since $x \mapsto x^2 - x - 1$ has its only positive zero at φ (Step 3) and is negative before it and positive after it, we have

$$k(x) > 0 \quad \text{for} \quad 0 < x < \varphi, \qquad\qquad k(x) < 0 \quad \text{for} \quad x > \varphi.$$

Now, $y_1 = x_1 = 1 < \varphi$ and $z_1 = x_2 = 1 + 1/1 = 2 > \varphi$. By induction, using that $h \circ h$ is increasing (Step 2) and fixes φ (Step 3): if $y_k < \varphi$ then $y_{k+1} = (h \circ h)(y_k) < (h \circ h)(\varphi) = \varphi$; hence $y_k < \varphi$ for every k. Symmetrically, $z_k > \varphi$ for every k. *Step 5 – monotonicity and convergence.* From $y_k < \varphi$ for every k and Step 4 it follows that $k(y_k) > 0$, i.e. $y_{k+1} > y_k$: (y_k) is strictly **increasing** and bounded above by φ. Symmetrically, (z_k) is strictly **decreasing** and bounded below by φ. By the monotone convergence theorem, both converge; by continuity of $h \circ h$, the two limits are fixed points of $h \circ h$, and by Step 3 the only fixed point available in $[1, 2]$ is φ:

$$y_k \to \varphi, \qquad\qquad z_k \to \varphi.$$

Step 6 – conclusion. The odd- and even-indexed subsequences of (x_n) both converge to φ: this suffices to conclude that the whole sequence converges, $x_n \to \varphi$. ■

Definition 5.2 (Fibonacci sequence). $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ *for* $n \geq 2$*:* $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

Proposition 5.2. $x_n = \dfrac{F_{n+1}}{F_n}$ *for every* $n \geq 1$.

Proof. By induction. Base case: $x_1 = 1 = F_2/F_1 = 1/1$. Induction step: suppose $x_{n-1} = F_n/F_{n-1}$. Then

$$x_n = 1 + \frac{1}{x_{n-1}} = 1 + \frac{F_{n-1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = \frac{F_{n+1}}{F_n},$$

using the Fibonacci definition $F_{n+1} = F_n + F_{n-1}$. ■ *Corollary.* Combining Proposition 5.1 with Proposition 5.2: $\lim\limits_{n\to\infty} \dfrac{F_{n+1}}{F_n} = \varphi$. **Connection with the ladder network (Section 1).** For $p = 1$, the recursive relation (1) is $r_n = 1 + \dfrac{r_{n-1}}{1 + r_{n-1}}$, with $r_1 = 2$.

Proposition 5.3. $r_n = \dfrac{F_{2n+1}}{F_{2n}}$ *for every* $n \geq 1$.

Proof. By induction. Base case: $n = 1$, $F_3/F_2 = 2/1 = 2 = r_1$. ✓. Induction step: suppose $r_{n-1} = F_{2n-1}/F_{2n-2}$. Then

$$r_n = 1 + \frac{F_{2n-1}/F_{2n-2}}{1 + F_{2n-1}/F_{2n-2}} = 1 + \frac{F_{2n-1}}{F_{2n-2} + F_{2n-1}} = 1 + \frac{F_{2n-1}}{F_{2n}} = \frac{F_{2n} + F_{2n-1}}{F_{2n}} = \frac{F_{2n+1}}{F_{2n}},$$

having used the Fibonacci definition twice ($F_{2n} = F_{2n-2} + F_{2n-1}$ and $F_{2n+1} = F_{2n} + F_{2n-1}$). ■ *Remark.* Note that $r_n = F_{2n+1}/F_{2n} = x_{2n}$: the sequence (r_n) of the ladder network is nothing

but the even-indexed subsequence of (x_n) – consistently, (z_k) in the proof of Proposition 5.1 was indeed decreasing, exactly as Theorem 2.4 had established for (r_n) using entirely independent arguments. Proposition 5.3, combined with Proposition 5.1, gives $R_{eq} = R\varphi$: Method 2 converges to the same result as Method 1 (Sections 2–3), via two independent routes.

6 Beyond the Golden Ratio: generalizations for arbitrary p

Notable numerical values of $r^* = R_{eq}/R = \dfrac{1+\sqrt{1+4p}}{2}$:

p	R_{eq}/R
1	$\varphi = \dfrac{1+\sqrt{5}}{2} \approx 1.618$
2	2
3	$\dfrac{1+\sqrt{13}}{2} \approx 2.303$
6	3
12	4
20	5

Remark. A pattern emerges: whenever $p = k(k+1)$ for $k = 1, 2, 3, \ldots$ (that is, $p = 2, 6, 12, 20, \ldots$, twice the triangular numbers), $1 + 4p = (2k+1)^2$ is a perfect square, and $R_{eq}/R = k + 1$ is an **exact integer**. Direct check: $1 + 4k(k+1) = 4k^2 + 4k + 1 = (2k+1)^2$.

Appendix: the infinite ladder network and the "metallic numbers"

The family of *metallic numbers* (or *metallic means*) was introduced by the Argentine mathematician Vera W. de Spinadel as a generalization of the Golden Ratio (de Spinadel, 1999 – see Bibliography). In its broadest formulation, the family comprises **all** positive quadratic irrational numbers that solve an equation of the form $x^2 - nx - 1 = 0$ **or** of the form $x^2 - x - n = 0$, for a positive integer n. *Careful not to confuse the two forms*: both belong to the family defined by Spinadel, but they give different numbers. The better-known form, $x^2 - nx - 1 = 0$, produces the popular sequence "Golden Ratio, Silver Ratio, Bronze Ratio, ..." ($n = 1 \Rightarrow \varphi$; $n = 2 \Rightarrow 1 + \sqrt{2} \approx 2.414$, the "silver ratio"; $n = 3 \Rightarrow \frac{3+\sqrt{13}}{2} \approx 3.303$, the "bronze ratio"). Our equation (3), rewritten in dimensionless form as $r^2 - r - p = 0$, is instead of the **second** form, $x^2 - x - n = 0$ with $n = p$. The two forms coincide only for $n = 1$ (where they both give φ): for $n = 2$, for instance, the first form gives the silver ratio ≈ 2.414, while ours (r^* for $p = 2$) gives exactly 2 – different numbers, both legitimately part of the same broader family, but not the same member. In short: the equivalent resistance of the ladder network, as a function of p, **is** a family of metallic numbers in Spinadel's sense (the $x^2 - x - n = 0$ branch), but it **is not** the more commonly cited popular "gold/silver/bronze" sequence – which belongs to the other branch of the same family.

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