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A Reciprocity Formula for Integrals of Symmetric Functions

Nihil sub sole novum

A functional symmetry condition that simplifies integrals over symmetric intervals

An in-depth guide for Calculus / Real Analysis courses

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by **Nicola Maglione** – keenonscience

@keenonmaths

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Introduction

Evaluating definite integrals of the form $\int_{-\alpha}^{\alpha} \Phi(x)\,dx$ often requires complex substitutions or the use of the Residue Theorem. However, when the integrand has specific symmetry properties, the computation simplifies considerably.

A classic case, common in mathematical competitions, is an integrand of the form $\Phi(x) = f(x)/(1+e^x)$ with f an even function: in this case, surprisingly, the integral over $[-\alpha, \alpha]$ coincides with the integral of f over the single interval $[0, \alpha]$. This article formalizes and extends this result to a broader class of functions $g(x)$, identifying the true structural property that enables the simplification: a **functional reciprocity** condition.

Section 1 states the theorem in its general form – replacing e^x with any function $g(x)$ satisfying $g(-x) = 1/g(x)$ – together with a remark clarifying where this condition comes from. Section 2 gives the complete proof, a direct computation based on splitting the integral and a change of variable. Section 3 applies the result to three concrete examples of increasing complexity, showing graphically the equivalence of the areas involved.

1 The Theorem

Let's fix the hypotheses once and for all. Let $f(x)$ be an **even** function, integrable on $[0, \alpha]$. Let $g(x)$ be a function such that

$$g(-x) = \frac{1}{g(x)},$$

with $g(x) \neq 0$ and $g(x) \neq -1$ for every $x \in [-\alpha, \alpha]$. We call this condition the **functional reciprocity** of g.

Theorem 1. *Under the hypotheses above, the following identity holds:*

$$\int_{-\alpha}^{\alpha} \frac{f(x)}{1+g(x)}\,dx = \int_{0}^{\alpha} f(x)\,dx.$$

In other words, the integral over the whole symmetric interval $[-\alpha, \alpha]$ of the composite integrand $f(x)/(1+g(x))$ coincides with the integral of f alone over half the interval – regardless of the specific shape of g, as long as it satisfies functional reciprocity. The case $\Phi(x) = f(x)/(1+e^x)$ mentioned in the introduction is simply the instance $g(x) = e^x$, which satisfies $g(-x) = e^{-x} = 1/g(x)$.

Observation 1. The condition $g(-x) = 1/g(x)$ implies that $\ln(g(x))$ is an **odd** function. Indeed, letting $h(x) = \ln(g(x))$,

$$h(-x) = \ln(g(-x)) = \ln\left(\frac{1}{g(x)}\right) = -\ln(g(x)) = -h(x).$$

Consequently, any function $g(x)$ of the form $e^{h(x)}$, with $h(x)$ odd, automatically satisfies the theorem's hypotheses. This is the most convenient way to build new examples:

pick any odd function $h(x)$ – such as x, x^3, $\sin x$, or a product like $x\cos x$ – and set $g(x) = e^{h(x)}$.

2 Proof

The proof is a direct computation, based on splitting the integral into the two sub-intervals $[-\alpha, 0]$ and $[0, \alpha]$ and a change of variable in the first one.

Proof. Split the integral into two parts:

$$\int_{-\alpha}^{\alpha} \frac{f(x)}{1+g(x)}\,dx = \int_{-\alpha}^{0} \frac{f(x)}{1+g(x)}\,dx + \int_{0}^{\alpha} \frac{f(x)}{1+g(x)}\,dx.$$

In the first term, multiply numerator and denominator by $1/g(x)$, and observe that, by hypothesis, $1/g(x) = g(-x)$:

$$\int_{-\alpha}^{0} \frac{f(x)}{1+g(x)}\,dx = \int_{-\alpha}^{0} \frac{f(x)\cdot g(-x)}{g(-x)+1}\,dx = \int_{-\alpha}^{0} \left(f(x) - \frac{f(x)}{1+g(-x)} \right) dx.$$

(The last step is pure algebra: $g(-x)/(g(-x)+1) = 1 - 1/(g(-x)+1)$.)

Substituting this back into the initial splitting, the total integral becomes

$$\int_{-\alpha}^{0} f(x)\,dx \;-\; \int_{-\alpha}^{0} \frac{f(x)}{1+g(-x)}\,dx \;+\; \int_{0}^{\alpha} \frac{f(x)}{1+g(x)}\,dx.$$

Now apply the change of variable $t = -x$ ($dt = -dx$) to the second integral: as x ranges over $[-\alpha, 0]$, t ranges over $[0, \alpha]$. Using also that f is even ($f(t) = f(-t)$):

$$\int_{0}^{\alpha} f(t)\,dt \;-\; \int_{0}^{\alpha} \frac{f(t)}{1+g(t)}\,dt \;+\; \int_{0}^{\alpha} \frac{f(x)}{1+g(x)}\,dx.$$

The second and third terms are the same integral (up to the name of the integration variable, which is irrelevant) with opposite sign: they cancel exactly, leaving

$$\int_{0}^{\alpha} f(x)\,dx,$$

which is what we wanted to prove. ∎

3 Examples

We now apply Theorem 1 to three concrete cases of increasing complexity. In each one we first check that g satisfies functional reciprocity, then apply the formula directly.

Example 1. Let $f(x) = x^2$ and $g(x) = e^x$. Since $g(-x) = e^{-x} = 1/e^x = 1/g(x)$, the condition is satisfied (this is the case already discussed in Observation 1, with $h(x) = x$). Theorem 1 gives:

$$\int_{-1}^{1} \frac{x^2}{1+e^x}\,dx = \int_{0}^{1} x^2\,dx = \frac{1}{3}.$$

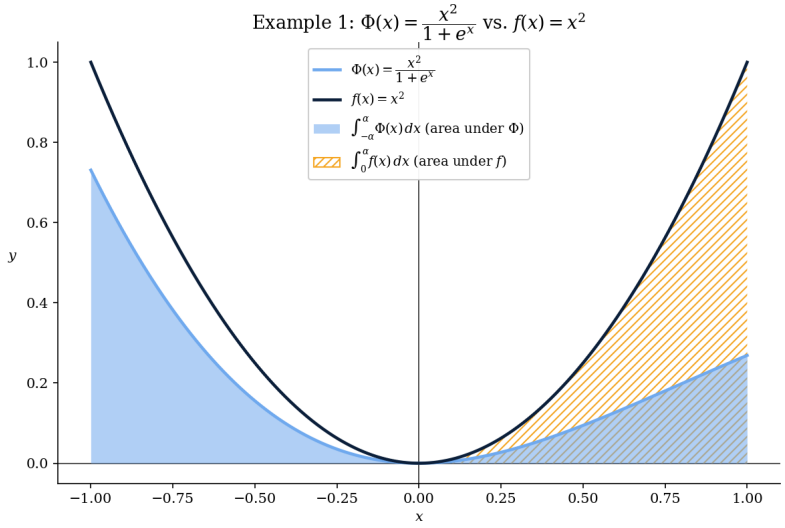

Figure 1: The blue area under $\Phi(x) = x^2/(1+e^x)$ on $[-1, 0]$ and the (hatched) area under $f(x) = x^2$ on $[0, 1]$ are equivalent: both measure $1/3$.

Example 2. Let $f(x) = x^2 + \cos x$ and $g(x) = e^{x\cos x}$. Here $h(x) = x\cos x$ is odd (the product of an odd function, x, and an even one, $\cos x$), so by Observation 1 the condition is automatically satisfied. Theorem 1 gives:

$$\int_{-\pi/2}^{\pi/2} \frac{x^2 + \cos x}{1 + e^{x\cos x}}\,dx = \int_{0}^{\pi/2} (x^2 + \cos x)\,dx = \frac{\pi^3}{24} + 1.$$

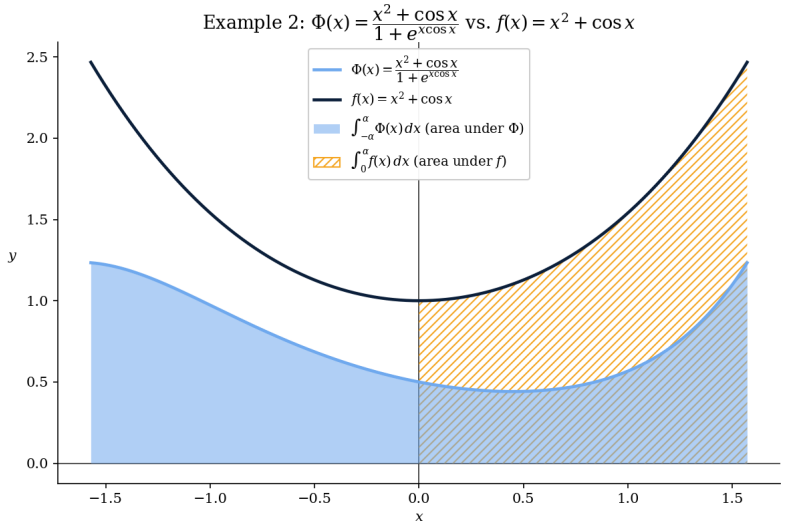

Figure 2: Same scheme for a more complex transcendental integrand: the two equivalent areas over $[-\pi/2, 0]$ and $[0, \pi/2]$.

Example 3. Let $f(x) = x^2$ and $g(x) = (1 + x^2)^x$. One checks directly that $g(-x) = (1 + (-x)^2)^{-x} = (1 + x^2)^{-x} = 1/g(x)$ (here g is not of the form $e^{h(x)}$ with h odd from Observation 1: this is an example of reciprocity verified directly, not through that shortcut). Theorem 1 gives:

$$\int_{-1}^{1} \frac{x^2}{1 + (1 + x^2)^x}\, dx = \int_{0}^{1} x^2\, dx = \frac{1}{3}.$$

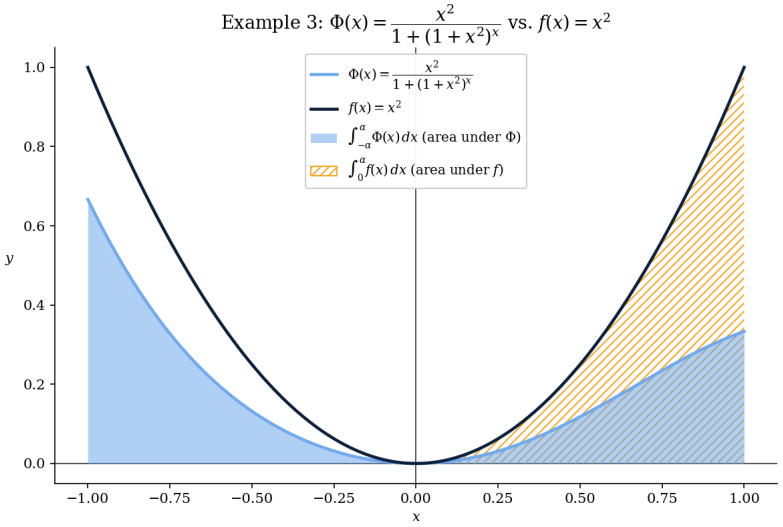

Figure 3: Application to an integrand built from a power function: here too the two equivalent areas measure 1/3.

4 Conclusion

The functional reciprocity of $g(x)$ – the condition $g(-x) = 1/g(x)$ – allows for a direct simplification of a whole class of integrals over symmetric intervals, reducing $\int_{-\alpha}^{\alpha} f(x)/(1 + g(x))\, dx$ to the simpler $\int_0^{\alpha} f(x)\, dx$. The case $\Phi(x) = f(x)/(1 + e^x)$, well known from mathematical competitions, is just the simplest instance of this more general phenomenon.

Observation 1 also provides a practical way to build new examples: simply pick any odd function $h(x)$ and set $g(x) = e^{h(x)}$. But, as Example 3 shows, functional reciprocity can also occur in forms that don't fit this pattern, and must then be verified directly.

In summary, this result provides an elegant tool for recognizing structural symmetries in integrands, avoiding otherwise laborious computations.

Bibliography

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About the author. Nicola Maglione is an electronic engineer (M.Sc. in Electronic Engineering, University of Naples, 2001), with over twenty years of experience in the semiconductor industry as a design and program manager, specializing in NAND Flash memories and high-speed interfaces, and holder of several patents in memory architectures. Alongside his professional career he has always cultivated a passion for mathematics – particularly Analysis, Abstract Algebra and Number Theory – and for physics, which he shares through keenonscience, his science outreach page, and on keenonmaths.org.

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